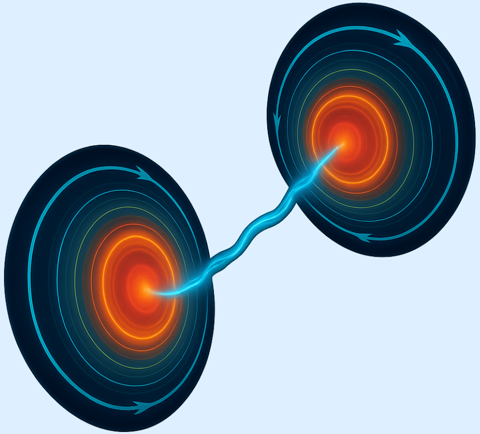

Anyon superconductivity in a Chern–Simons–Landau–Ginzburg theory of the fractional quantum Hall effect

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Motivation

Motivation

- ZHK model provides an effective field theory for the fractional quantum Hall effect[1]
- Describes the system as a condensate of anyons coupled to a Chern–Simons gauge field
- We extend that idea to a relativistic setting

⇒ Vortices act as flux–charge composites with superconducting behavior[2]

- Previous studies introduce auxiliary neutral scalar field to make the model self-dual[3]
- Others consider a generalization with a scalar-field-dependent dielectric function[4], or both[5]

⇒ BPS and non-interacting vortices

- We explicitly do not do this and study **non-BPS** vortex anyons
- We consider the Chern–Simons–Landau–Ginzburg, or Maxwell–Chern–Simons–Higgs, model

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[1] S. C. Zhang, T. H. Hansson, and S. Kivelson, Phys. Rev. Lett. 62, 82 (1989)
[2] D.-H. Lee and M. P. A. Fisher, Phys. Rev. Lett. 63, 903 (1989)
[3] G. V. Dunne and C. A. Trugenberger, Rev. D 43, 1323 (1991)
[4] P. K. Ghosh, Phys. Rev. D 49, 5458 (1994)
[5] J. Andrade, R. Casana, and E. da Hora, Phys. Rev. D 111, 036019 (2025)

Chern–Simons–Landau–Ginzburg theory

Model setup and parameters

- Superconducting order parameter / condensate / Higgs field $\psi : \mathbb{R}^{2+1} \to \mathbb{C}$
- Abelian gauge field $A = (A_0, \mathbf{A}) \in \mathbb{R}^{2+1}$
- Gauge covariant derivative $D_\mu = \partial_\mu + iqA_\mu$
- Gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

$\Rightarrow$ Magnetic field $B = F_{12}$ and electric field $E_i = F_{0i}$

- Minkowski spacetime $\mathbb{R}^{2+1}$, endowed with metric η and signature $(+--)$

CSLG theory

- CSLG theory is a topologically massive gauge theory that is Lorentz invariant
- The CSLG Lagrangian is[6,7]

$$\mathcal{L} = \frac{1}{2}D_\mu\psi\overline{D^\mu\psi} - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} - V(|\psi|) + \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma}$$

- First three terms correspond to the Ginzburg–Landau, or abelian Higgs, model
- Last term is the topological Chern–Simons term

$$\mathcal{L}_{\mathrm{CS}} = \frac{\kappa}{4}\epsilon^{\alpha\beta\gamma}A_\alpha F_{\beta\gamma} = \frac{\kappa}{2}\left(A_0 B - \epsilon_{ij}A_i E_j\right)$$

- CS term breaks parity (P) and time reversal (T) explicitly, but preserves PT
- We are interested in the effect the CS term has on vortices

[6] T. Hansson, V. Oganesyan, and S. Sondhi, Ann. Phys. 313, 497 (2004)

[7] E. Fradkin, Phys. Rev. B 42, 570 (1990)

Gauss' law

- Static Lagrangian of the CSLG theory is

$$\mathcal{L}_{\text{static}} = \frac{1}{2}(\partial_i A_0)^2 + \frac{\kappa}{2}\left(A_0 B + \epsilon_{ij} A_i \partial_j A_0\right) + \frac{1}{2} q^2 A_0^2 |\psi|^2 - \left[\frac{1}{2} D_i\psi \overline{D_i\psi} + \frac{1}{2}B^2 + V(|\psi|)\right]$$

- Can simplify by an integration by parts

$$\int_{\mathbb{R}^2} \mathrm{d}^2x\, \epsilon_{ij} A_i \partial_j A_0 = \int_{\mathbb{R}^2} \mathrm{d}^2x\, A_0 B$$

- Hence, static Lagrangian can be expressed as

$$\mathcal{L}_{\text{static}} = \frac{1}{2}(\partial_i A_0)^2 + \kappa A_0 B + \frac{1}{2} q^2 A_0^2 |\psi|^2 - \left[\frac{1}{2} D_i\psi \overline{D_i\psi} + \frac{1}{2}B^2 + V(|\psi|)\right]$$

- Varying this w.r.t. A_0 reveals Gauss' law as an elliptic PDE

$$\frac{\delta \mathcal{L}_{\text{static}}}{\delta A_0} = 0 \quad \Rightarrow \quad \left(-\nabla^2 + q^2|\psi|^2\right) A_0 = -\kappa B$$

Maxwell charge

- Gauss' law enforces that electric charge and magnetic flux are not independent
- The electric field is $\mathbf{E} = -\boldsymbol{\nabla} A_0 \neq \mathbf{0}$
- Compute electric charge density via Maxwell equation & Gauss law

$$\left(-\nabla^2 + q^2|\psi|^2\right) A_0 = -\kappa B \qquad \Rightarrow \qquad \rho_e = \boldsymbol{\nabla} \cdot \mathbf{E} = -\nabla^2 A_0 = -\kappa B - q^2|\psi|^2 A_0$$

- Total electric charge is

$$Q_e = \int_{\mathbb{R}^2} \mathrm{d}^2x\, \rho_e = -\kappa\Phi - q^2 \int_{\mathbb{R}^2} \mathrm{d}^2x\, A_0|\psi|^2, \qquad \Phi = \int_{\mathbb{R}^2} \mathrm{d}^2x B$$

- Localized static solutions: A_0 decays exponentially and $E_i \to 0$ at spatial infinity

$\Rightarrow$ $Q_e = 0$ for localized solutions, and

$$\int_{\mathbb{R}^2} \mathrm{d}^2x\, q^2 A_0|\psi|^2 = -\kappa\Phi$$

Flux-charge binding

- Condensate carries nontrivial internal $U(1)$ charge Q_m
- Associated Noether (super)current is

$$J_\mu = \frac{iq}{2}(\psi\partial_\mu\bar{\psi} - \bar{\psi}\partial_\mu\psi) + q^2 A_\mu|\psi|^2$$

- Corresponding Noether matter charge density

$$\rho_m = J_0 = q^2 A_0|\psi|^2$$

- Magnetic flux Φ and Noether charge Q_m are bound together by[8]

$$\int_{\mathbb{R}^2} d^2x\, q^2 A_0|\psi|^2 = -\kappa\Phi \qquad \Rightarrow \qquad Q_m = \int_{\mathbb{R}^2} d^2x\, \rho_m = -\kappa\Phi$$

$\Rightarrow$ Each vortex simultaneously carries a flux quantum Φ and a proportional electric charge $Q_m = -\kappa\Phi$

[8] S. Deser, R. Jackiw, and S. Templeton, Ann. Phys. 281, 409 (2000)

Flux-charge binding is topological

- The flux-charge binding mechanism is purely topological
- Arises from the Chern–Simons term and Gauss law, Maxwell term plays no part
- Consider the Chern–Simons–Higgs model[9]

$$\mathcal{L} = \frac{1}{2} D_\mu \psi \overline{D^\mu \psi} + \frac{\kappa}{4} \epsilon^{\alpha\beta\gamma} A_\alpha F_{\beta\gamma} - V(|\psi|)$$

- Gauss' law is algebraic

$$A_0 = -\frac{\kappa B}{q^2 |\psi|^2}$$

- Noether charge is still

$$Q_m = \int_{\mathbb{R}^2} \mathrm{d}^2 x \, q^2 A_0 |\psi|^2 = -\kappa \int_{\mathbb{R}^2} \mathrm{d}^2 x B = -\kappa \Phi$$

- This is the flux–charge binding mechanism $\rightarrow$ purely topological, enforced by the CS term

[9] S. A. Parameswaran, S. A. Kivelson, E. H. Rezayi, S. H. Simon, S. L. Sondhi, and B. Z. Spivak, Phys. Rev. B 85, 241307 (2012)

Vortex anyons

Static energy

- The static energy of the CSLG theory is

$$\mathcal{E} = -\mathcal{L}_{\text{static}} = \frac{1}{2}D_i\psi\overline{D_i\psi} + \frac{1}{2}B^2 + V(|\psi|) - \frac{1}{2}(\partial_i A_0)^2 - \kappa A_0 B - \frac{1}{2}q^2 A_0^2|\psi|^2$$

- At a first glance this appears not to be bounded from below
- Inner product of Gauss' law with the A_0 and integrating by parts gives

$$-\int_{\mathbb{R}^2} \mathrm{d}^2x\, \kappa A_0 B = \int_{\mathbb{R}^2} \mathrm{d}^2x \left[-A_0\partial_i\partial_i A_0 + q^2|\psi|^2 A_0^2\right] = \int_{\mathbb{R}^2} \mathrm{d}^2x \left[(\partial_i A_0)^2 + q^2|\psi|^2 A_0^2\right]$$

- Using this relation yields an energy that is positive (semi-)definite and bounded below

$$E = \int_{\mathbb{R}^2} \mathrm{d}^2x \left\{\frac{1}{2}|\mathbf{D}\psi|^2 + \frac{1}{2}B^2 + V(|\psi|) + \frac{1}{2}|\boldsymbol{\nabla} A_0|^2 + \frac{1}{2}q^2 A_0^2|\psi|^2\right\}$$

Non-local $\rightarrow$ constrained local

- Static vortex anyons are minimizers of the static energy

$$E = \int_{\mathbb{R}^2} \mathrm{d}^2x \left\{ \frac{1}{2}|\mathbf{D}\psi|^2 + \frac{1}{2}B^2 + V(|\psi|) + \frac{1}{2}|\boldsymbol{\nabla} A_0|^2 + \frac{1}{2}q^2 A_0^2 |\psi|^2 \right\}$$

subject to Gauss' law

$$\left(-\nabla^2 + q^2|\psi|^2\right) A_0 = -\kappa B$$

- As the energy is bounded below $\Rightarrow$ amenable to minimization methods
- This is inherently a non-local problem, but has been reformulated as a constrained minimization problem
- Challenges of this type are not unique to the CSLG framework
- They arise more broadly across both condensed matter and high energy physics

Non-local problems in high energy & condensed matter physics

- Nuclear skyrmions stabilized by ω-mesons[10,11] (Skyrme field - $\varphi \in SU(2)$, potential - $\omega \in \mathbb{R}$):

$$\mathcal{E} = \frac{1}{8}\,|d\varphi|^2 + \frac{1}{4}V(\varphi) + \frac{1}{2}\,|d\omega|^2 + \frac{1}{2}\omega^2, \qquad \left(-\nabla^2 + 1\right)\omega = -c_\omega \mathcal{B}_0$$

- Demagnetization in chiral magnets[12] (Magnetization - $\mathbf{n} \in S^2$, magnetic potential - $\psi \in \mathbb{R}$):

$$\mathcal{E} = \frac{J}{2}|d\mathbf{n}|^2 + \mathcal{D}\sum_{i=1}^{3}\mathbf{d}_i \cdot (\mathbf{n} \times \partial_i \mathbf{n}) + V(\mathbf{n}) + \frac{1}{2\mu_0}|d\psi|^2, \qquad -\nabla^2\psi = -\mu_0 M_s[\boldsymbol{\nabla} \cdot \mathbf{n}]$$

- Flexoelectric self-polarization in chiral liquid crystals[13] (Director - $\mathbf{n} \in \mathbb{R}P^2$, electric potential - $\varphi \in \mathbb{R}$):

$$\mathcal{E} = \frac{K}{2}|d\mathbf{n}|^2 + Kq_0\left[\mathbf{n} \cdot (\boldsymbol{\nabla} \times \mathbf{n})\right] + V(\mathbf{n}) + \frac{\epsilon_0}{2}|d\varphi|^2, \qquad -\nabla^2\varphi = -\frac{1}{\epsilon_0}[\boldsymbol{\nabla} \cdot \mathbf{P}_f(\mathbf{n})]$$

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[10] S. B. Gudnason and M. Speight, J. High Energ. Phys. 07, 184 (2020)

[11] D. Harland, P. Leask, and M. Speight, J. High Energ. Phys. 06, 116 (2024)

[12] P. Leask and M. Speight, arXiv:2504.17772 [cond-mat.mes-hall]

[13] P. Leask, Phys. Rev. Res. 7, 043001 (2025)

Constrained Newton flow

- Problems of this nature are well-suited to the **constrained Newton flow** method[14]
- Static vortex anyons are critical points of the static energy

⇒ Solutions of static Ginzburg–Landau equations

$$D_i D_i \psi = 2\frac{\partial V}{\partial \bar{\psi}} - q^2 A_0^2 \psi, \qquad \partial_j(\partial_j A_i - \partial_i A_j) = J_i - \kappa \epsilon_{ij} \partial_j A_0$$

- Must also satisfy Gauss' law $\left(-\partial_i \partial_i + q^2 |\psi|^2\right) A_0 = -\kappa B$
- We reformulate Gauss constraint as an unconstrained optimization problem

$$\min_{\psi \text{ const.}} F(A_0), \qquad F(A_0) = \int_{\mathbb{R}^2} \mathrm{d}^2 x \left\{ \frac{1}{2} |\nabla A_0|^2 + \frac{1}{2} q^2 |\psi|^2 A_0^2 + \kappa B A_0 \right\}$$

- Solve using non-linear conjugate gradient descent with line search strategy
- Conjugate step-size updated using Polak–Ribière–Polyak method

[14] The CUDA code for this method in the CSLG theory is available on my public github repository https://github.com/Paulnleask/cuSuperAnyon

Constrained Newton flow

- We now solve the static GL equations, assuming A_0 satisfies the Gauss constraint

$\Rightarrow$ Arrested Newton flow[15]: Accelerated gradient descent method with flow arresting criteria

- Formulate the minimization as a second order dynamical problem and solve the second order system

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}t^2} = \frac{1}{2}D_iD_i\psi - \frac{\partial V}{\partial\bar{\psi}} + \frac{1}{2}q^2A_0^2\psi, \qquad \frac{\mathrm{d}^2A_i}{\mathrm{d}t^2} = \partial_j(\partial_jA_i - \partial_iA_j) - J_i + \kappa\epsilon_{ij}\partial_jA_0,$$

- Can be reduced to a coupled first order system $\Rightarrow$ solve using RK4
- As initial configuration, we use an extended Abrikosov–Nielsen–Olesen (ANO) ansatz[16,17]

$$\psi = m\phi(r)e^{iN\theta}, \qquad \mathbf{A} = \frac{Na(r)}{qr}\left(\sin\theta, -\cos\theta\right), \qquad A_0 = -\frac{\kappa B}{q^2m^2}$$

[15] S. B. Gudnason and J. M. Speight, J. High Energ. Phys. 07, 184 (2020)
[16] A. Abrikosov, J. Phys. Chem. Solids. 2, 199 (1957)
[17] H. B. Nielsen and P. Olesen, Nucl. Phys. B 61, 45 (1973)

Hybrid superconducting typology

Abelian Higgs model

- Consider AH model with the conventional quartic Higgs potential

$$\mathcal{L} = \frac{1}{2} D_\mu \psi \overline{D^\mu \psi} - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} - V(|\psi|), \qquad V(|\psi|) = \frac{\lambda}{8}\left(m^2 - |\psi|^2\right)^2$$

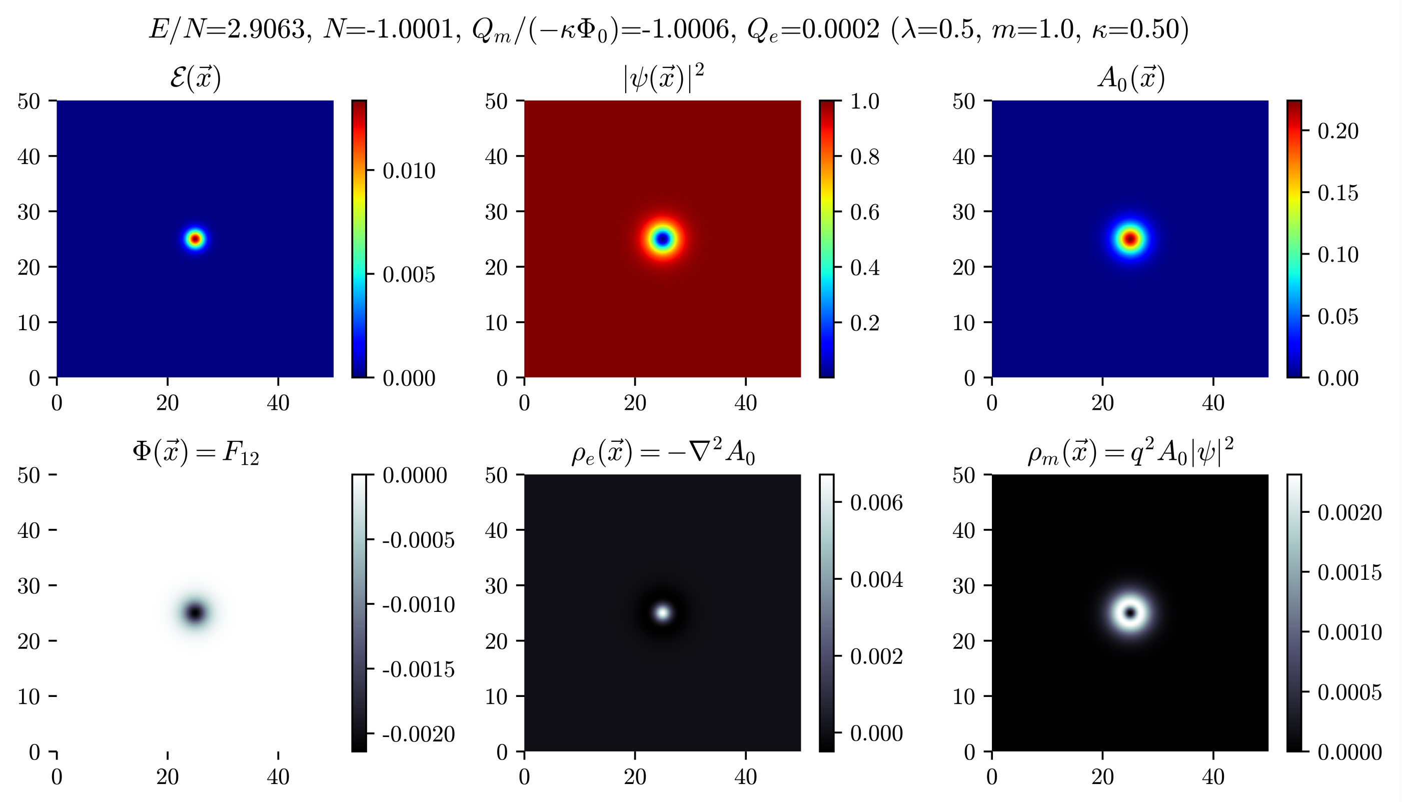
- Higgs mass $m_H = \sqrt{\lambda} m$, coherence length $\xi_s = 1/m_H$
- Proca mass $m_A = qm$, magnetic penetration depth $\xi_m = 1/m_A$
- GL parameter dictating superconducting typology is $\kappa_{\mathrm{GL}} = \xi_m/\xi_s = \sqrt{\lambda}/q$:
  - $\sqrt{\lambda} < q$: Type-I, attractive intervortex force
  - $\sqrt{\lambda} > q$: Type-II, repulsive intervortex force
  - $\sqrt{\lambda} = q$: BPS, no intervortex force

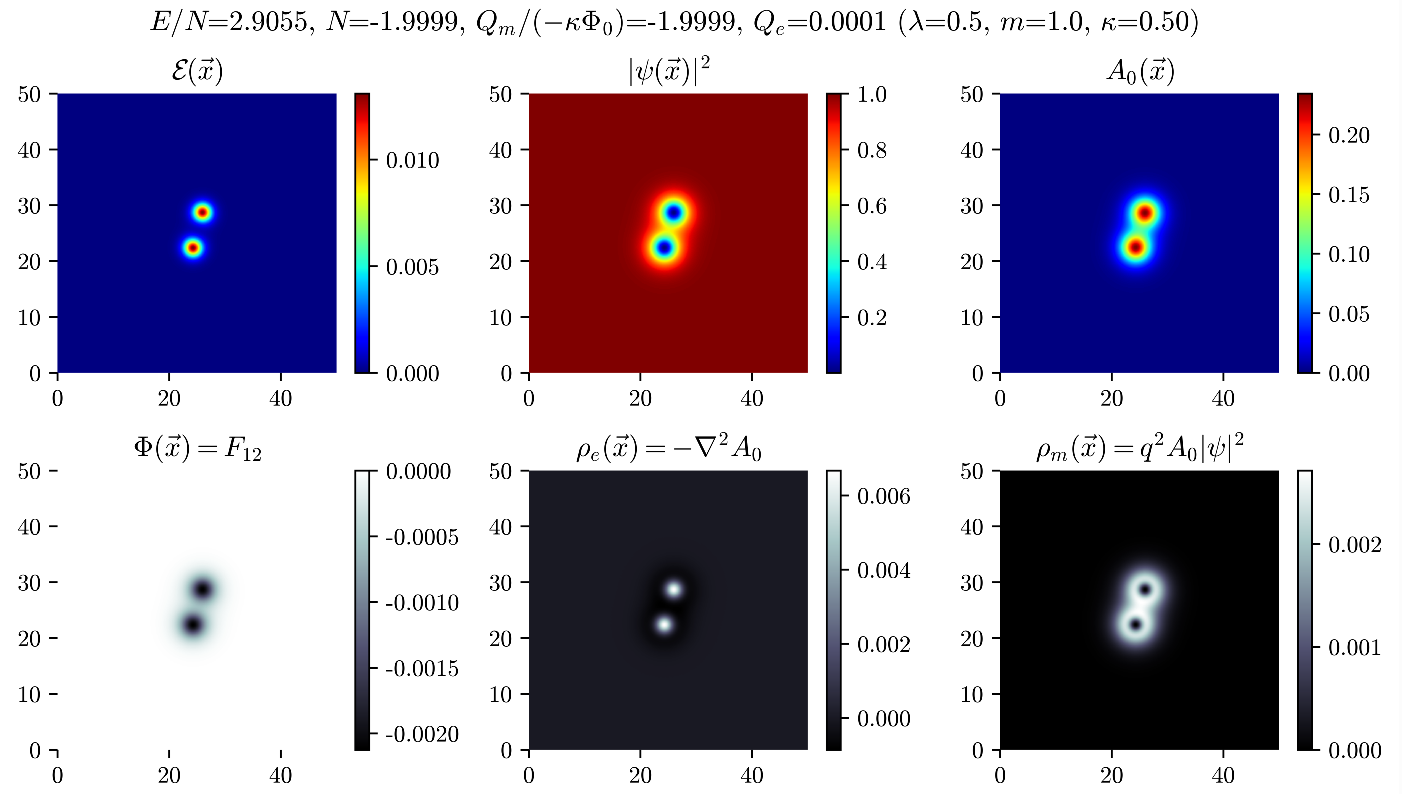
Anyon bound states

- CSLG theory: type-I/II dichotomy is broken
- Each vortex carries a magnetic flux + proportional Noether electric charge

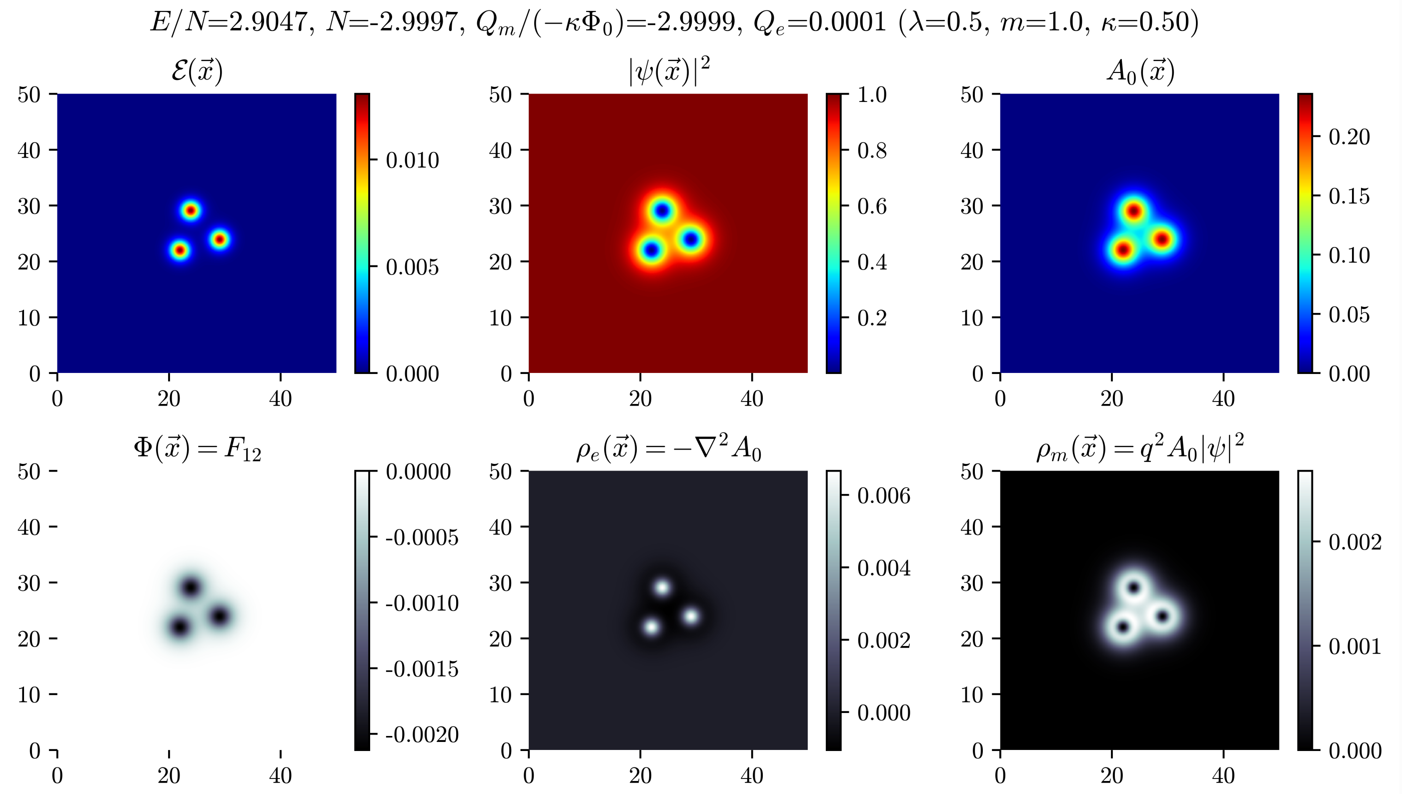
$\Rightarrow$ Induces electrostatic repulsion between vortices

- In the typical type-II repulsive regime ($\kappa > 1$), the repulsive interaction force is now stronger
- At critical coupling $\lambda = 1$, vortices now repel one another
- In the type-I attractive regime ($\lambda < 1$), κ can force vortex cores (zeros of ψ) to split
- If κ is large, the interaction force becomes entirely repulsive
- For relatively small κ, vortex cores split but remain bounded


E/N=2.9063, N=-1.0001, $Q_m/(-\kappa\Phi_0)$=-1.0006, Q_e=0.0002 (λ=0.5, m=1.0, κ=0.50)
$\mathcal{E}(\vec{x})$
$|\psi(\vec{x})|^2$
$A_0(\vec{x})$
$\Phi(\vec{x}) = F_{12}$
$\rho_e(\vec{x}) = -\nabla^2 A_0$
$\rho_m(\vec{x}) = q^2 A_0 |\psi|^2$



E/N=2.9055, N=-1.9999, $Q_m/(-\kappa\Phi_0)$=-1.9999, Q_e=0.0001 (λ=0.5, m=1.0, κ=0.50)
$\mathcal{E}(\vec{x})$
$|\psi(\vec{x})|^2$
$A_0(\vec{x})$
$\Phi(\vec{x}) = F_{12}$
$\rho_e(\vec{x}) = -\nabla^2 A_0$
$\rho_m(\vec{x}) = q^2 A_0 |\psi|^2$
0.010
0.005
0.000
1.0
0.8
0.6
0.4
0.2
0.20
0.15
0.10
0.05
0.00
0.0000
-0.0005
-0.0010
-0.0015
-0.0020
0.006
0.004
0.002
0.000
0.002
0.001
0.000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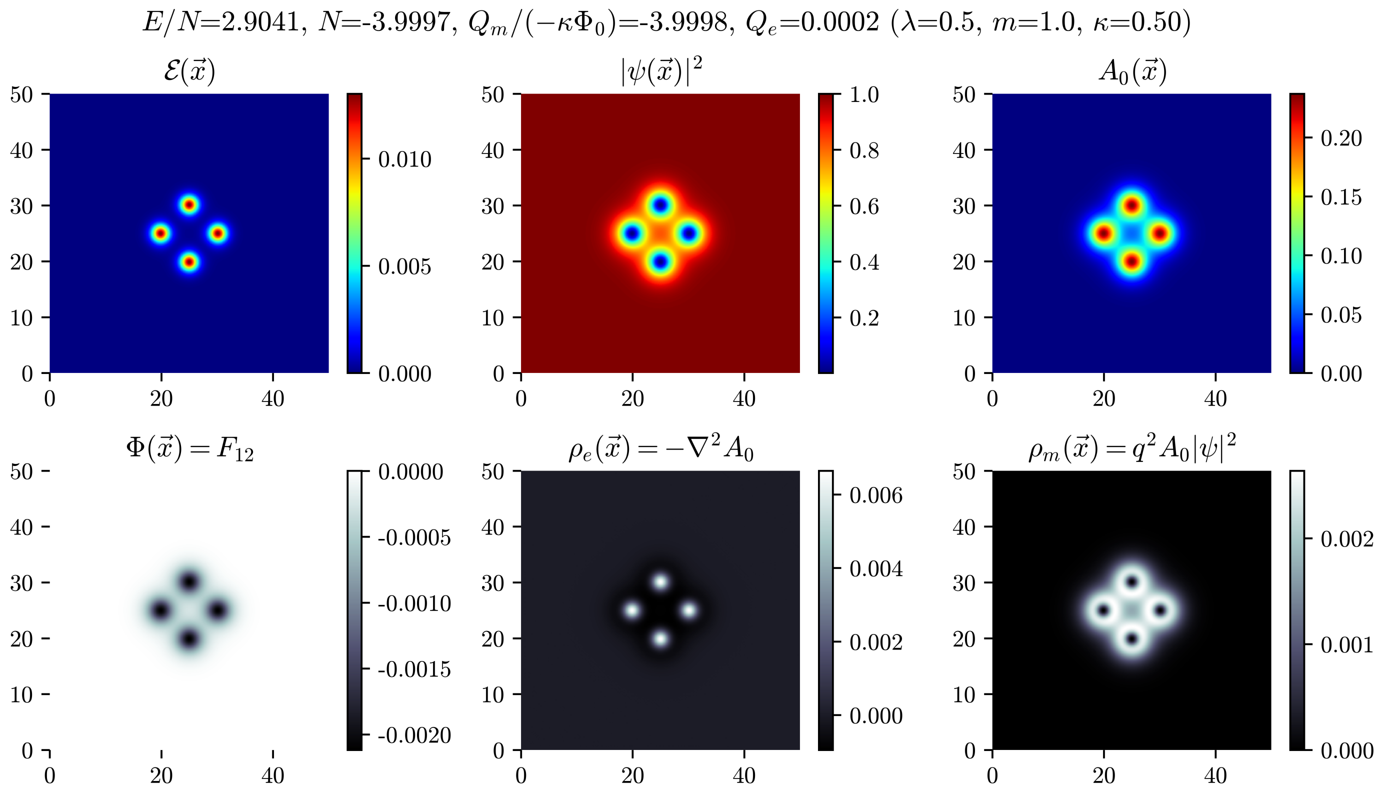
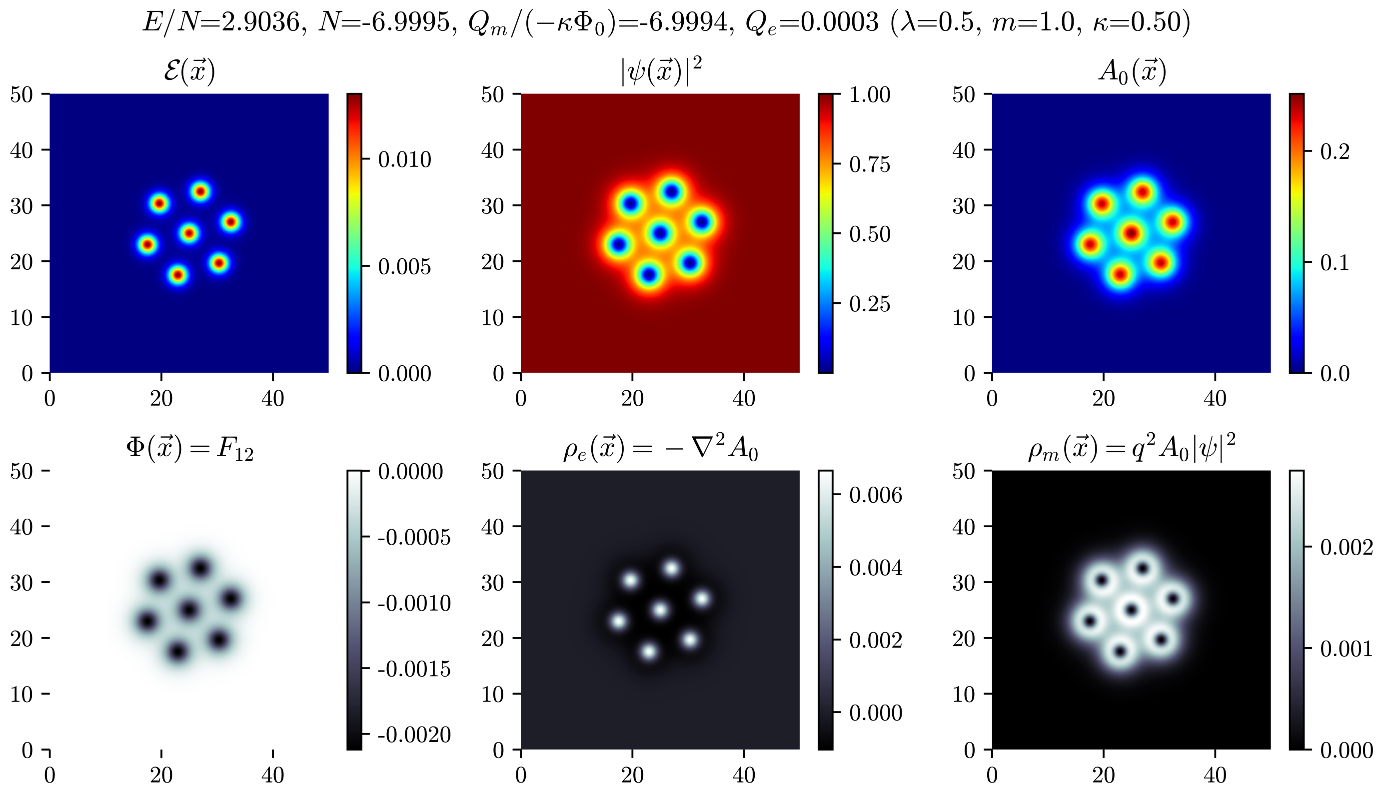
$\mathcal{E}(\vec{x})$
$|\psi(\vec{x})|^2$
$A_0(\vec{x})$
$\Phi(\vec{x}) = F_{12}$
$\rho_e(\vec{x}) = -\nabla^2 A_0$
$\rho_m(\vec{x}) = q^2 A_0 |\psi|^2$


E/N=2.9041, N=-3.9997, $Q_m/(-\kappa\Phi_0)$=-3.9998, Q_e=0.0002 (λ=0.5, m=1.0, κ=0.50)
$\mathcal{E}(\vec{x})$
$|\psi(\vec{x})|^2$
$A_0(\vec{x})$
$\Phi(\vec{x}) = F_{12}$
$\rho_e(\vec{x}) = -\nabla^2 A_0$
$\rho_m(\vec{x}) = q^2 A_0 |\psi|^2$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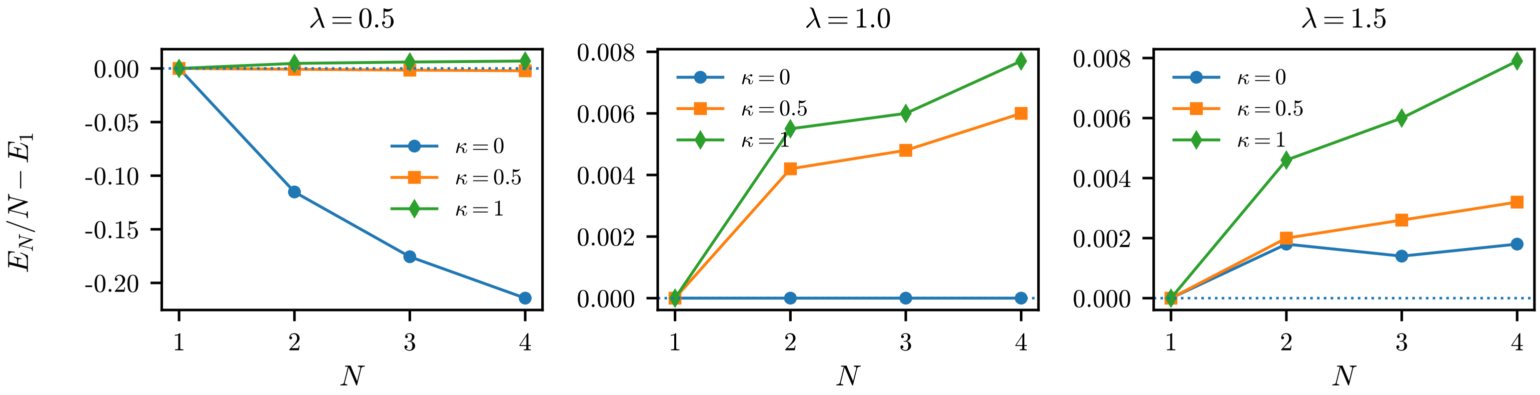
$|\psi(\vec{x})|^2$
$A_0(\vec{x})$
$\Phi(\vec{x}) = F_{12}$
$\rho_e(\vec{x}) = -\nabla^2 A_0$
$\rho_m(\vec{x}) = q^2 A_0 |\psi|^2$
0.000
0.005
0.010
0.25
0.50
0.75
1.00
0.0
0.1
0.2
0.0000
-0.0005
-0.0010
-0.0015
-0.0020
0.000
0.002
0.004
0.006
0.000
0.001
0.002

Binding energies



Hybrid superconductivity

- Consider the type-I regime ($\lambda < 1$) with κ large enough to cause core splitting
- Binding energy remains negative and interaction energy is non-monotonic

$\Rightarrow$ Bound stable multi-vortex anyon states

$\Rightarrow$ Hybridization of type I/II superconductivity behavior

- Look to understand long-range interactions for clues
- Must first consider the static screening structure and penetration depths

Screening structure

Dynamical gauge masses

- Let us focus on the gauge field by considering the abelian Maxwell–Chern–Simons Lagrangian

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\kappa}{4}\epsilon^{\mu\nu\rho}A_\mu F_{\nu\rho} + \frac{1}{2}m_A^2 A_\mu A^\mu$$

- $m_A = qm$ is the usual Higgs (Proca) mass from symmetry breaking
- Equation of motion for A_μ (in the Lorenz gauge $\partial_\mu A^\mu = 0$) after Fourier transforming $\partial_\mu \mapsto ip_\mu$ is

$$\left[(p^2 - m_A^2)\eta_{\mu\nu} + i\kappa\epsilon_{\mu\nu\rho}p^\rho\right] A^\nu = 0, \qquad p = (\omega, \mathbf{k})$$

- Inverse propagator in momentum space (Green's operator) is

$$\mathcal{D}^{-1}_{\mu\nu}(p) = (p^2 - m_A^2)\eta_{\mu\nu} + i\kappa\epsilon_{\mu\nu\rho}p^\rho$$

- For a massive excitation at rest ($\mathbf{k} = \mathbf{0}$), the dynamical gauge masses are the propagator poles

$$\det \mathcal{D}^{-1}(\omega, \mathbf{k} = \mathbf{0}) = 0 \quad \Rightarrow \quad \omega_\pm^2 = M_\pm^2, \qquad M_\pm = \sqrt{m_A^2 + \frac{\kappa^2}{4}} \pm \frac{\kappa}{2}$$

Dynamical gauge masses $M_\pm = \sqrt{m_A^2 + \kappa^2/4} \pm \kappa/2$

- $M_\pm$ are physical masses of propagating gauge excitations (topologically massive photons)
- They describe two physical propagating modes with different masses and helicities
- CS term assigns a handedness (chirality) to the gauge field, with helicities having differing masses

$\Rightarrow$ Breaks parity and time reversal (each reverses handedness), but combination PT restores it

- Parity breaking "splits" B^μ into two on-shell masses $M_\pm$[18]
- These have associated length scales $l_\pm = 1/M_\pm$
- Paul–Khare[19] identifies $l_\pm$ as penetration depths with l_- giving rise to an energetically favorable vortex
- Consistency check: abelian Higgs limit $\lim_{\kappa\to 0} M_\pm = m_A$ and $\lim_{\kappa\to 0} l_\pm = \lambda = 1/m_A$ ✓

[18] R. D. Pisarski and S. Rao, Phys. Rev. D 32, 2081 (1985)

[19] S. K. Paul and A. Khare, Phys. Lett. B 174, 420 (1986)

Static far-field asymptotics

- Alternatively: can also obtain static screening masses and penetration depths by considering static long-range asymptotics
- Let us work in the unitary gauge $\psi \in \mathbb{R}$ and the Coulomb gauge $\partial_i A_i = 0$
- Linearize about ground state $\{\psi, A_\mu\} = \{m + \phi, 0 + a_\mu\}$
- Higgs field has mass $m_H = \sqrt{V''(m)} = m\sqrt{\lambda}$ and Proca mass is $m_A = qm$
- Slight abuse of notation: $B = \epsilon_{ij}\partial_i a_j$ and $E_i = -\partial_i a_0$
- Static energy, linearized about ground state, is

$$E_{\text{lin}} = \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{d}^2x \left[\phi\left(-\nabla^2 + m_H^2\right)\phi\right] + \frac{1}{2}\int_{\mathbb{R}^2} \mathrm{d}^2x \begin{bmatrix} B & a_0 \end{bmatrix} \begin{bmatrix} \left(-\nabla^2 + m_A^2\right) & -\kappa\nabla^2 \\ \kappa & \left(-\nabla^2 + m_A^2\right) \end{bmatrix} \begin{bmatrix} B \\ a_0 \end{bmatrix}$$

Static far-field asymptotics

- To linear order, Gauss constraint and static Ginzburg–Landau equations reduce to

$$\left(-\nabla^2 + m_H^2\right)\phi = 0, \quad \left(-\nabla^2 + m_A^2\right) a_i = \kappa\epsilon_{ij}\partial_j a_0, \quad \left(-\nabla^2 + m_A^2\right) a_0 = -\kappa B$$

- Higgs-amplitude mode ϕ decouples giving a static Klein-Gordon equation
- Taking the curl of the linearized gauge field equation yields

$$(-\nabla^2 + m_A^2)B = \kappa\nabla^2 a_0 \qquad (*)$$

- Applying the Laplace operator to the linearized Gauss' law gives

$$(-\nabla^2 + m_A^2)\nabla^2 a_0 = -\kappa\nabla^2 B \qquad (**)$$

- Applying the operator $(-\nabla^2 + m_A^2)$ to (*) and using relation (**), gives a scalar decoupled fourth order equation for the magnetic field

$$\left[\left(-\nabla^2 + m_A^2\right)^2 + \kappa^2\nabla^2\right] B = 0$$

Static screening masses

- Magnetic field and electric field satisfy same linearized field equations, $\Delta_\kappa B = 0$ and $\Delta_\kappa E_i = 0$, where

$$\Delta_\kappa = \left[\left(-\nabla^2 + m_A^2\right)^2 + \kappa^2\nabla^2\right] = \left(\nabla^2 - m_+^2\right)\left(\nabla^2 - m_-^2\right)$$

- Takes same form as linearized field equation for OP in superfluids with fermionic imbalance[20]
- Δ_κ can be factorized into **complex**-conjugate eigenmodes $m_\pm$ (static screening masses)[21]

$$m_\pm = \sqrt{m_A^2 - \frac{\kappa^2}{4}} \pm i\frac{\kappa}{2}$$

- These masses do not agree with our computation of the dynamical gauge masses $M_\pm \in \mathbb{R}$...

[20] M. Barkman, A. Samoilenka, T. Winyard, and E. Babaev, Phys. Rev. Res. 2, 043282 (2020)

[21] M. Stålhammar, D. Rudneva, T. H. Hansson, and F. Wilczek, Phys. Rev. B 109, 064514 (2024)

Penetration depths: dynamical gauge or static screening?

- Dynamical gauge masses $M_\pm \in \mathbb{R}$, whereas static screening masses $m_\pm \in \mathbb{C}$, with

$$M_\pm = \sqrt{m_A^2 + \kappa^2/4} \pm \kappa/2, \qquad m_\pm = \sqrt{m_A^2 - \kappa^2/4} \pm i\kappa/2$$

- Why the discrepancy and which masses define the penetration depths?
- In both cases, the abelian Higgs limit is recovered $\lim_{\kappa\to 0} m_\pm = \lim_{\kappa\to 0} M_\pm = m_A$ and $\lim_{\kappa\to 0} \lambda_\pm = \lim_{\kappa\to 0} l_\pm = \lambda$
- **Dynamical gauge** masses are the poles of the propagator in **Minkowski** space $D_M(\omega, \mathbf{0})$
- **Static screening** masses are the poles of the propagator in **Euclidean** space $D_E(0, \mathbf{k})$
- We must have analytic continuation between Minkowski and Euclidean formulations

$\Rightarrow$ Consistency condition ensuring that Euclidean and Minkowski propagators describe the same analytic structure

Penetration depths: static screening masses ✓

- Dynamical gauge masses $M_{\pm} \in \mathbb{R}$, whereas static screening masses $m_{\pm} \in \mathbb{C}$, with

$$M_{\pm} = \sqrt{m_A^2 + \kappa^2/4} \pm \kappa/2, \qquad m_{\pm} = \sqrt{m_A^2 - \kappa^2/4} \pm i\kappa/2$$

- To be self-consistent as a QFT they must be related by a **Wick rotation** $p^0 = \omega \mapsto i\omega$
- This translates to an **effective** Wick rotation $\kappa \mapsto i\kappa$
- Complex-conjugate static poles correspond to imaginary continuation of the real-time propagator poles to Euclidean frequency axis
- In AH model, dynamical gauge masses are identical to static screening masses
- CS term breaks parity and this is no longer true
- Penetration depths (screening lengths) are related to static screening masses, not dynamical gauge masses

Dynamical gauge ($M_\pm, l_\pm$) vs static screening ($m_\pm, \lambda_\pm$)

- $M_\pm$: tell you how fast the gauge field oscillates in time $\rightarrow$ *dynamical response*
- $l_\pm$: propagation length scales or Compton wavelengths
- $m_\pm$: tell you how fast the fields decay in space $\rightarrow$ *static screening*
- $\lambda_\pm$: spatial structure of static fields (e.g. vortex profiles)
- Static screening masses are

$$m_\pm = \alpha \pm i\beta, \quad \alpha = \sqrt{m_A^2 - \kappa^2/4}, \quad \beta = \frac{\kappa}{2}$$

$\Rightarrow$ Magnetic & electric fields share common penetration depth λ_{gauge} but differ by oscillation frequency $1/\lambda_{\text{osc}}$

$$\lambda_{\text{gauge}} = \frac{1}{\alpha} = \frac{1}{\sqrt{m_A^2 - \kappa^2/4}}, \qquad \lambda_{\text{osc}} = \frac{2\pi}{\beta} = \frac{4\pi}{\kappa}.$$

Long-range interactions

Long-range interactions

- Can determine long-range interactions following point-particle method[22]
- Done in two parts:
  1. Add linear sources to linearized energy, such that solutions of field equations are exactly single-vortex far fields
  2. Compute the interaction energy from the on-shell cross term in the linearization
- After a bit of work[23], the interaction energy of a pair of separated vortex anyons is given by

$$V_{\mathrm{int}}(R) \simeq 2\pi \left[|c_B|^2 e^{-\alpha R} \cos\left(\beta R - \gamma\right) - c_H^2 K_0(m_H R) \right]$$

- Standard Higgs contribution remains monotone attractive
- Gauge term becomes a damped oscillator with envelope $e^{-\alpha R}/\sqrt{R}$, decay rate $\alpha = \sqrt{m_A^2 - \kappa^2/4}$, and oscillation frequency $\beta = \kappa/2$, alternating between attractive and repulsive behavior

[22] J. M. Speight, Phys. Rev. D 55, 3830 (1997)
[23] P. Leask, arXiv:2510.04830 [cond-mat.supr-con]

Long-range interactions

- Oscillatory attractive/repulsive behavior of gauge contribution leads to non-monotonic interactions
- If gauge term is dominant over Higgs term at long-range:
  - Provides a repulsive force initially ($R < \frac{\pi}{\kappa} + \frac{2\gamma}{\kappa}$)
  - Switches to an attractive force at longer range ($R > \frac{\pi}{\kappa} + \frac{2\gamma}{\kappa}$)
  - Repeats this behavior in a decaying oscillatory fashion as the R increases
- Breaks usual vanilla type I/II dichotomy

⇒ Hybrid of type I & II superconductivity behavior

- Similar behavior arises in multiband superconductors, called type 1.5 superconductivity[24,25]
- Hybrid behavior there arises to due competing length scales with $\xi_1 < \lambda < \xi_2$
- Hybrid behavior here arises from the decaying oscillatory behavior of the gauge field

[24] E. Babaev and M. Speight, Phys. Rev. B 72, 180502 (2005)

[25] E. Babaev, J. Carlström, and M. Speight, Phys. Rev. Lett. 105, 067003 (2010)

Abelian Higgs limit

- Consistency check, must recover AH model in the limit $\kappa \to 0$
- Decay rate becomes $\lim_{\kappa\to 0}\alpha = m_A = qm$ and oscillatory behavior vanishes, $\lim_{\kappa\to 0}\beta = 0$

$\Rightarrow$ Magnetic penetration depth is recovered

$$\lim_{\kappa\to 0}\lambda_{\text{gauge}} = \frac{1}{\alpha} = \frac{1}{m_A}$$

- Complex-conjugate screening masses tend to single real-valued Proca mass

$$\lim_{\kappa\to 0} m_\pm = \alpha = m_A$$

- Also recover long-range interaction energy of AH model[26,27]

$$\lim_{\kappa\to 0} V_{\text{int}}(R) = 2\pi\left[c_B^2 K_0(m_A R) - c_H^2 K_0(m_H R)\right]$$

[26] L. M. A. Bettencourt and R. J. Rivers, Phys. Rev. D 51, 1842 (1995)

[27] K. Fujikura, S. Li, and M. Yamaguchi, J. High Energ. Phys. 12, 115 (2023)

Conclusion and further work

Conclusion

- Gauss' law binds magnetic flux to electric charge $\Rightarrow$ anyonic vortices
- CS term makes screening masses complex
- Electric and magnetic fields decay with common penetration depth but acquire oscillatory phase shift
- Breaks type-I/II dichotomy $\Rightarrow$ new hybrid typology
- Vortex anyons form stable bound states with separated cores

$\Rightarrow$ Theoretical realization of hybrid superconducting behavior in an anyon superconductor

- Future directions include:
  - Systematic study of vortex lattice phases[28]
  - Role of the CS term in dynamical interactions of vortex anyons[29]
  - Short-range interactions of vortex anyons[30]

[28] M. Speight and T. Winyard, J. Phys. A: Math. Theor. 58, 095203 (2025)
[29] D. Bazeia, J. G. F. Campos, and A. Mohammadi, J. High Energ. Phys. 12, 108 (2024)
[30] M. Speight and T. Winyard, Phys. Rev. D 112, 055024 (2025)