Magnetostatic self-interaction effect on bulk magnetic skyrmion textures in chiral ferromagnets

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Motivation

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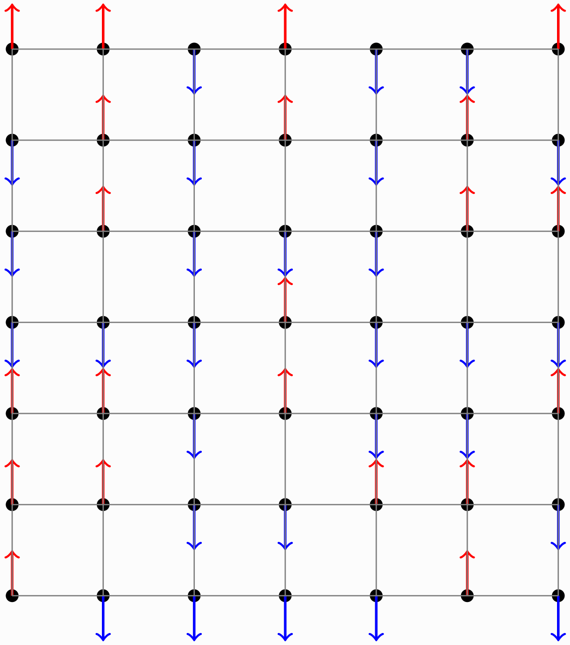
- The magnetization in a ferromagnet has a dipolar moment associated with it
- This induces an internal demagnetizing magnetic field
- In turn, this generates a magnetostatic self-energy
- Want to find topological solitons (static energy minimizers)

$\Longrightarrow$ There is back-reaction from the self-induced magnetic field on the magnetization

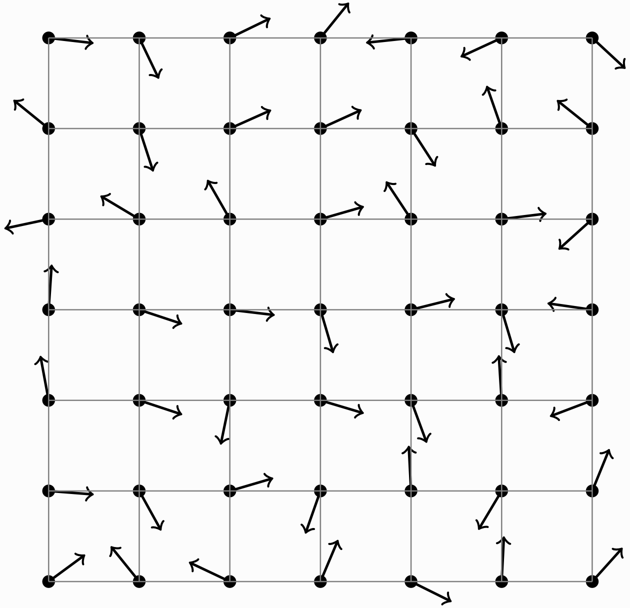
- How can this back-reaction be determined?

Static energy of the model

Ising vs Heisenberg



(a) Ising



(b) Heisenberg

Heisenberg magnet

- Consider the Heisenberg model of a system of spins $\vec{S}_i$ that are localised on the sites of a d-dimensional lattice:

$$H_H = -\sum_{i,j} \mathbb{J}_{ij} \vec{S}_i \cdot \vec{S}_j, \quad \mathbb{J}_{ij} = \begin{cases} J & \text{if } i, j \text{ are neighbours} \\ 0 & \text{else} \end{cases}$$

- Mean field approximation: a proper dynamical variable is the expectation value of the spins, i.e. the unit magnetization $\vec{n} \in S^2 \subset \mathbb{R}^3$
- Heisenberg Hamiltonian becomes

$$H_H = -J \sum_{i,j} \vec{n}_i \cdot \vec{n}_{i+a\vec{e}_j},$$

where $a\vec{e}_j$ is the vector connecting a lattice site i with its neighbouring sites $i + a\vec{e}_j$

Heisenberg magnet

- Taylor expansion of $\vec{n}_{i+a\vec{e}_j}$ for small lattice constant a gives

$$\vec{n}_i \cdot \vec{n}_{i+a\vec{e}_j} = \underbrace{\vec{n}_i \cdot \vec{n}_i}_{=1} + a \underbrace{\left(\vec{n}_i \cdot \partial_j \vec{n}_i\right)}_{=0} + \frac{a^2}{2} \vec{n}_i \cdot \partial_j^2 \vec{n}_i + O(a^3)$$

- Ignoring the constant $\vec{n}_i \cdot \vec{n}_i = 1$, the Heisenberg Hamiltonian in the continuum limit is

$$H_H \approx -\frac{Ja^2}{2} \sum_{i,j} \vec{n}_i \cdot \partial_j^2 \vec{n}_i \quad \rightarrow \quad E_{\text{exch}}[\vec{n}] = -\frac{Ja^{2-d}}{2} \int_{\mathbb{R}^d} \mathrm{d}^d x \left(\vec{n} \cdot \partial_j^2 \vec{n}\right)$$

- Integration by parts gives us the static energy of the $O(3)$ sigma model

$$E_{\text{exch}}[\vec{n}] = \frac{Ja^{2-d}}{2} \int_{\mathbb{R}^d} \mathrm{d}^d x \left(\partial_j \vec{n} \cdot \partial_j \vec{n}\right), \quad \vec{n} \cdot \vec{n} = 1$$

Dzyaloshinksii–Moriya interaction

- At the lattice level, the DMI (an antisymmetric exchange interaction) is [Sov. Phys. JETP **19**, 960 (1964)]

$$H_D = \mathcal{D} \sum_{i,j} \vec{d}_j \cdot \left(\vec{n}_i \times \vec{n}_{i+a\vec{e}_j} \right)$$

- Taylor expanding again gives

$$\vec{n}_i \times \vec{n}_{i+a\vec{e}_j} = \underbrace{\vec{n}_i \times \vec{n}_i}_{=\vec{0}} + a\vec{n}_i \times \partial_j \vec{n}_i + O(a^2)$$

- DMI in the continuum limit is

$$H_D \approx \mathcal{D}a \sum_{i,j} \vec{d}_j \cdot \left(\vec{n}_i \times \partial_j \vec{n}_i \right) \quad \rightarrow \quad E_{\mathrm{DMI}}[\vec{n}] = \mathcal{D}a^{1-d} \int_{\mathbb{R}^d} \mathrm{d}^d x \sum_i \vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n})$$

- Topological spin textures arise due to competition between Heisenberg exchange interaction and DMI
- Heisenberg exchange energy promotes parallel alignment of spins whereas DMI favors non-collinear alignment of spins (spin canting)

Continuum energy

- The energy in the continuum limit (for $d = 3$) is $E = E_{\text{exch}} + E_{\text{DMI}} + E_{\text{pot}}$ where [Zh. Eksp. Teor. Fiz. **95**, 178-182 (1989)]

$$E_{\text{exch}} = \frac{J}{2} \int_{\mathbb{R}^3} \mathrm{d}^3 x \, |\mathrm{d}\vec{n}|^2,$$

$$E_{\text{DMI}} = \mathcal{D} \int_{\mathbb{R}^3} \mathrm{d}^3 x \sum_{i=1}^{3} \vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n}),$$

$$E_{\text{pot}} = \int_{\mathbb{R}^3} \mathrm{d}^3 x \left(K_m (1 - n_z^2) + M_s B_{\text{ext}} (1 - n_z) \right),$$

- Skyrmions are **topological solitons** in this model [e.g. New J. Phys. **18**, 065003 (2016)]
- These are static solutions to the Euler-Lagrange field equations
- Chiral magnet can also be derived from string theory [J. High Energ. Phys. **11**, 212 (2023)]

Magnetic skyrmions

Topological magnetic spin textures

- Ground state configuration found by minimizing $E_{\text{pot}} \implies \vec{n}_{\text{vac}} = \vec{n}_{\uparrow} = (0, 0, 1)$
- Finite (static) energy requires $\vec{n}(\vec{x}) \to \vec{n}_{\uparrow}$ as $|\vec{x}| \to \infty$

$\implies$ One-point compactification of space $\mathbb{R}^2 \cup \{\infty\} \cong S^2$

- Magnetization $\vec{n}$ is effectively a based map $\vec{n} : S^2 \mapsto S^2$
- Gives rise to a non-trivial homotopy group $\pi_2(S^2) = \mathbb{Z}$
- Configuration space is seen to consist of disconnected manifolds,

$$M = \bigcup_{i \in \mathbb{Z}} M_i$$

- Magnetization configurations are identified by the topological degree

$$\text{deg}(\vec{n}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \text{d}^2x \, \vec{n} \cdot \left(\frac{\partial \vec{n}}{\partial x_1} \times \frac{\partial \vec{n}}{\partial x_2} \right) \in \pi_2(S^2) = \mathbb{Z}$$

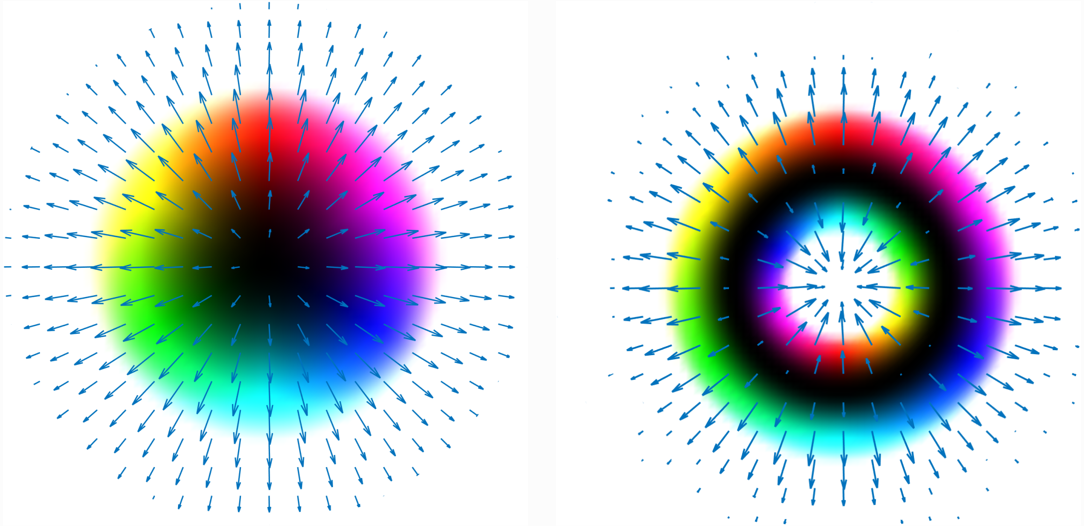
Magnetic vortices

- Magnetic vortices are **minimizers** of the static energy functional
- Consider the radially symmetric ansatz

$$\vec{n}(r,\theta) = (\sin f(r)\cos\theta, \sin f(r)\sin\theta, \cos f(r))$$

- Reduces problem to solving a non-linear ODE in $f(r)$
- Profile function $f(r)$ decreases monotonically with boundary conditions $f(0) = k\pi$ and $f(\infty) = 0$
- This gives us $\vec{n}(r=0) = \vec{n}_{\downarrow}$ and $\vec{n}(r=\infty) = \vec{n}_{\uparrow}$
- Gives rise to $k\pi$-vortices
- $k = 1$ yields a π-vortex, more commonly known as a **Néel skyrmion** with $\deg(\vec{n}) = -1$
- $k = 2$ yields a 2π-vortex, known as **skyrmionium** with $\deg(\vec{n}) = 0$ [J. Magn. Magn. Mater. **195**, 182-192 (1999)]

Magnetic vortices



(a) Néel skyrmion (b) Skyrmionium

Figure: The magnetization $\vec{n} = (n_1, n_2, n_3) \in S^2$ is coloured using the Runge colour sphere. Spin-up states $\vec{n}_\uparrow = (0, 0, 1)$ are white, whereas spin-down states $\vec{n}_\downarrow = (0, 0, -1)$ are black. The hue is determined by the phase $\arg(n_1 + in_2)$.

Magnetic skyrmions

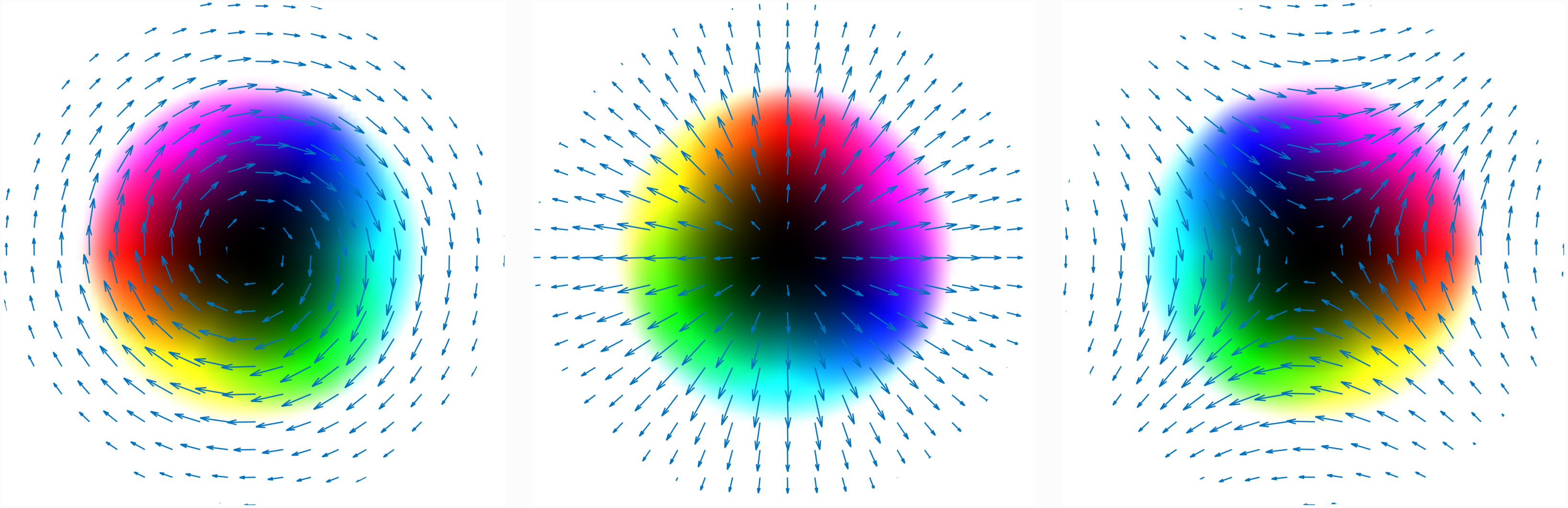
- Other radially symmetric ansätze can be considered

$\Longrightarrow$ Different vortex/skyrmion solutions

- Skyrmion type is actually determined by the DMI term used
- We consider three different DMI terms: [Lifshitz invariants $\Lambda_{ij}^{(k)} = n_i \frac{\partial n_j}{\partial x_k} - n_j \frac{\partial n_i}{\partial x_k}$]

	Dresselhaus	**Rashba**	**Heusler**
$\{\vec{d}_1, \vec{d}_2\}$	$\{-\vec{e}_1, \vec{e}_2\}$	$\{\vec{e}_2, -\vec{e}_1\}$	$\{\vec{e}_1, -\vec{e}_2\}$
DMI	$\Lambda_{xz}^{(y)} - \Lambda_{yz}^{(x)}$	$\Lambda_{zx}^{(x)} - \Lambda_{yz}^{(y)}$	$\Lambda_{xz}^{(y)} + \Lambda_{yz}^{(x)}$
Skyrmion	Bloch	Néel	Antiskyrmion
$\deg(\vec{n})$	-1	-1	$+1$
$\vec{n}$	$\begin{pmatrix} -\sin f(r) \sin\theta \\ \sin f(r) \cos\theta \\ \cos f(r) \end{pmatrix}$	$\begin{pmatrix} \sin f(r) \cos\theta \\ \sin f(r) \sin\theta \\ \cos f(r) \end{pmatrix}$	$\begin{pmatrix} -\sin f(r) \sin\theta \\ -\sin f(r) \cos\theta \\ \cos f(r) \end{pmatrix}$

Skyrmion types



(a) Bloch (b) Néel (c) Antiskyrmion

Magnetostatic self-energy and its back-reaction

Magnetostatic problem

- The magnetization behaves as a magnetic dipole
- This induces an **internal** magnetic field (demagnetizing field)

$\Longrightarrow$ Generates a magnetostatic self-energy

- Computing this energy and the back-reaction on the magnetization is difficult
- Dzyaloshinksii–Moriya interaction assumed to be more significant
- Stripe domain structures in ordinary ferromagnets with no interfacial DMI were studied analytically [Phys. Rev. B **48**, 10335 (1993)]
- Analytic investigation including dipolar interactions for Dresselhaus DMI with Néel type modulations [Phys. Rev. Lett. **105**, 197202 (2010)]
- Numerical study of dipolar interaction with Monte-Carlo simulations [J. Magn. Magn. Mater **324**, 2171 (2012)]

Self-induced magnetic field

- The magnetic field induced by an isolated dipole of moment $\vec{m}$ at $\vec{0}$ is

$$\vec{B} = -\frac{\mu_0}{4\pi r^3}\left(\vec{m} - 3\frac{\vec{m}\cdot\vec{x}}{r^2}\vec{x}\right)$$

- This can be expressed as the gradient of a magnetic potential $\psi : \mathbb{R}^3 \to \mathbb{R}$, that is,

$$\vec{B} = -\vec{\nabla}\psi, \quad \psi = -\mu_0\vec{m}\cdot\vec{\nabla}\left(\frac{1}{4\pi r}\right)$$

- Consider now a continuous distribution of magnetic dipole density $\vec{m} : \Omega \to \mathbb{R}^3$, where $\Omega \subseteq \mathbb{R}^3$ is some domain. The magnetic field it induces, at a point $\vec{x} \in \mathbb{R}^3$, is given by integrating the field induced at $\vec{x}$ by $\vec{m}(\vec{y})$ at $\vec{y} \in \Omega$ over $\vec{y} \in \Omega$:

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi}\int_\Omega \mathrm{d}^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|^3}\left\{\vec{m}(\vec{y}) - 3\frac{\vec{m}(\vec{y})\cdot(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^2}(\vec{x}-\vec{y})\right\}$$

The magnetic potential

- Again, this is the gradient of the magnetic potential $\psi : \Omega \to \mathbb{R}$,

$$\vec{B} = -\vec{\nabla}\psi, \quad \psi(\vec{x}) = -\mu_0 \int_\Omega \mathrm{d}^3\vec{y} \left\{ \vec{m}(\vec{y}) \cdot \vec{\nabla}_x \left(\frac{1}{4\pi|\vec{x}-\vec{y}|} \right) \right\}$$

- We now note that the Green's functions for the Laplacian $\Delta = -\nabla^2$ on $\Omega \subseteq \mathbb{R}^3$ is

$$G(\vec{x},\vec{y}) = \frac{1}{4\pi|\vec{x}-\vec{y}|} \quad \Longrightarrow \quad \Delta_x G(\vec{x},\vec{y}) = \delta(\vec{x}-\vec{y})$$

- The Laplacian of the potential is

$$\begin{aligned} \Delta_x \psi(\vec{x}) &= -\mu_0 \int_\Omega \mathrm{d}^3\vec{y} \left\{ \vec{m}(\vec{y}) \cdot \vec{\nabla}_x \left[\Delta_x G(\vec{x},\vec{y}) \right] \right\} \\ &= -\mu_0 \int_\Omega \mathrm{d}^3\vec{y} \left\{ \delta(\vec{x}-\vec{y}) \left[\vec{\nabla}_y \cdot \vec{m}(\vec{y}) \right] \right\} \\ &= -\mu_0 \vec{\nabla}_x \cdot \vec{m}(\vec{x}) \end{aligned}$$

Interaction energy of a distribution of magnetic dipoles

- Therefore, we see that the magnetic potential ψ satisfies Poisson's equation

$$\Delta\psi = \mu_0\rho, \quad \rho = -\left(\vec{\nabla}\cdot\vec{m}\right)$$

- The magnetic field induced by the dipole distribution $\vec{m}$ coincides, therefore, with the electric field induced by the charge distribution $-\left(\vec{\nabla}\cdot\vec{m}\right)$ [Phys. Rev. B **20**, 33 (1979)]
- Can think of $-\left(\vec{\nabla}\cdot\vec{m}\right)$ as "electric charge density"
- The interaction energy of a pair of magnetic dipoles $\vec{m}^{(1)}$, $\vec{m}^{(2)}$, is $-\vec{m}^{(1)}\cdot\vec{B}^{(2)}$, where $\vec{B}^{(2)}$ is the magnetic field induced by $\vec{m}^{(2)}$ at the position of $\vec{m}^{(1)}$
- Hence, the total dipole-dipole interaction energy of a continuous dipole density distribution is

$$E_{\mathrm{DDI}} = -\frac{1}{2}\int_{\Omega} \mathrm{d}^3\vec{x}\,\left(\vec{B}(\vec{x})\cdot\vec{m}(\vec{x})\right)$$

DDI energy = electrostatic self-energy

- It is useful to rewrite the DDI energy as a functional of the magnetic potential ψ:

$$
\begin{aligned}
E_{\text{DDI}} &= \frac{1}{2}\int_\Omega \mathrm{d}^3\vec{x}\,\vec{m}\cdot\vec{\nabla}\psi \\
&= -\frac{1}{2}\int_\Omega \mathrm{d}^3\vec{x}\,\left(\vec{\nabla}\cdot\vec{m}\right)\psi + \frac{1}{2}\int_{\partial\Omega} \mathrm{d}\vec{s}\cdot\left(\psi\vec{m}\right) \\
&= \frac{1}{2\mu_0}\int_\Omega \mathrm{d}^3\vec{x}\,\psi\Delta\psi + \frac{1}{2}\int_{\partial\Omega} \mathrm{d}\vec{s}\cdot\left(\psi\vec{m}\right)
\end{aligned}
$$

- We will use this formula only in situations where the boundary conditions ensure that the boundary term vanishes. In this case, E_{DDI} coincides with the "electrostatic" self-energy of the "charge" distribution $-(\nabla\cdot\vec{m})$
- To see this, we use the general identity $\psi\Delta\phi = \vec{\nabla}\psi\cdot\vec{\nabla}\phi - \vec{\nabla}\cdot\left(\psi\vec{\nabla}\phi\right)$ and the divergence theorem to express the DDI energy as

$$
E_{\text{DDI}} = \frac{1}{2\mu_0}\int_\Omega \mathrm{d}^3\vec{x}\,|\vec{\nabla}\psi|^2 = \frac{1}{2\mu_0}\int_\Omega \mathrm{d}^3\vec{x}\,|\vec{B}|^2
$$

Interaction energy summary

- Can compute the electrostatic self-energy in a few ways, most computationally expensive is:

$$E_{\mathrm{DDI}} = -\frac{1}{2}\int_\Omega \mathrm{d}^3\vec{x}\,\left(\vec{B}(\vec{x})\cdot\vec{m}(\vec{x})\right), \quad \vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi}\int_\Omega \mathrm{d}^3\vec{y}\frac{1}{|\vec{x}-\vec{y}|^3}\left\{\vec{m}(\vec{y}) - 3\frac{\vec{m}(\vec{y})\cdot(\vec{x}-\vec{y})}{|\vec{x}-\vec{y}|^2}(\vec{x}-\vec{y})\right\}$$

- Better to determine the magnetic vector potential $\psi : \Omega \to \mathbb{R}$ first, such that

$$E_{\mathrm{DDI}} = \frac{1}{2\mu_0}\int_\Omega \mathrm{d}^3\vec{x}\,\left(\psi(\vec{x})\Delta_x\psi(\vec{x})\right)$$

- You can choose to do it non-locally (computationally expensive)

$$\psi(\vec{x}) = -\mu_0\int_\Omega \mathrm{d}^3\vec{y}\,\left\{\vec{m}(\vec{y})\cdot\vec{\nabla}_x\left(\frac{1}{4\pi|\vec{x}-\vec{y}|}\right)\right\}$$

- This non-locality can be avoided by instead introducing the constraint that the magnetic potential ψ must satisfy the Poisson equation $\Delta_x\psi(\vec{x}) = -\mu_0\left(\vec{\nabla}_x\cdot\vec{m}(\vec{x})\right)$

Interaction energy summary

- The magnetization $\vec{m}$ behaves like a magnetic dipole moment

$\Longrightarrow$ Induces an internal demagnetizing magnetic field $\vec{B}$

- Want to include the magnetostatic self-energy $\mathcal{E}_{\mathrm{DDI}} = |\vec{B}|^2/2$

$\Longrightarrow$ Minimization problem $\rightarrow$ **constrained** minimization problem $\mathcal{E}_{\mathrm{DDI}} = \frac{1}{2\mu_0}\psi\Delta\psi$, where $\Delta\psi = -\mu_0\vec{\nabla}\cdot\vec{m}$

- Also want to determine the back-reaction of $\vec{B}$ on $\vec{m}$

$\Longrightarrow$ Need to compute the variation of $\mathcal{E}_{\mathrm{DDI}}$ w.r.t. $\vec{m}$

Reduction to planar magnetic skyrmions

Translation invariant solutions

- Case of interest: dipole-dipole interaction energy of a chiral ferromagnet in a translation invariant configuration
- We impose translation invariance in the direction $\vec{e}_3 = (0, 0, 1)$, so $\vec{n}$ is independent of x_3
- Consider fields $\vec{n} : \mathbb{R}^2 \longrightarrow S^2$ which have compact support in the sense that there exists $R_0 > 0$ such that, for all $r := |(x_1, x_2)| \geq R_0$, $\vec{n}(x_1, x_2) = \vec{n}_\uparrow$
- Since $\vec{n}$ is translation invariant, total dipole interaction energy either vanishes (for example, if $\vec{n}$ is constant) or diverges
- The energy per unit length (in the $\vec{e}_3$ direction) may be finite however
- Coincides with the total energy of the slab $\Omega = \mathbb{R}^2 \times [0, 1]$, which we compute as the limit of the energy of the thick disk $\Omega_R = \{\vec{x} \ : \ x_1^2 + x_2^2 \leq R^2, \ 0 \leq x_3 \leq 1\}$ as $R \rightarrow \infty$

Boundary conditions

- Boundary term $\mathrm{d}\vec{s}\cdot(\psi\vec{m})$ vanishes identically for all $R > R_0$
- Hence

$$E_{\mathrm{DDI}} = \frac{1}{2\mu_0}\int_{\mathbb{R}^2} \mathrm{d}^2x\,\psi\Delta\psi, \quad \Delta\psi = -\mu_0 M_s\left(\frac{\partial n_1}{\partial x_1} + \frac{\partial n_2}{\partial x_2}\right)$$

- Consider the large r behaviour of $\psi : \mathbb{R}^2 \to \mathbb{R}$
- Any solution of the Poisson equation $\Delta\psi = \rho$ on $\mathbb{R}^2$ has a multipole expansion

$$\psi = -\frac{q}{2\pi}\log r + O(r^{-1}), \quad q = \int_{\mathbb{R}^2} \mathrm{d}^2x\,\rho$$

- In general, such functions are logarithmically unbounded
- In our case ρ is (proportional to) the divergence of the in-plane field (n_1, n_2), so $q = 0$ by the divergence theorem

$\Longrightarrow$ ψ is (at least) $1/r$ localized

Variation of the magnetostatic self-energy

Variation of the dipolar energy

- Let $\vec{n}_t$ be a smooth variation of $\vec{n} = \vec{n}_0$ and define $\vec{\epsilon} = \partial_t \vec{n}_t|_{t=0}$
- Denote ψ_t the associated solution of $\Delta\psi_t = -\mu_0 \mathcal{M}_s \vec{\nabla}\cdot\vec{n}_t$ decaying to 0 at infinity, and $\dot{\psi} = \partial_t \psi_t|_{t=0}$
- The variation of E_{DDI} induced by $\vec{n}_t$ is

$$
\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E_{\mathrm{DDI}}(\vec{n}_t) &= \frac{1}{2\mu_0}\int_{\mathbb{R}^2} \mathrm{d}^2x \left(\dot{\psi}\Delta\psi + \psi\Delta\dot{\psi}\right) \\
&= \frac{1}{\mu_0}\int_{\mathbb{R}^2} \mathrm{d}^2x\, \psi\Delta\dot{\psi} + \lim_{R\to\infty}\frac{1}{2\mu_0}\int_{\partial B_R(0)} (\dot{\psi} \star \mathrm{d}\psi - \psi \star \mathrm{d}\dot{\psi}) \\
&= \frac{1}{\mu_0}\int_{\mathbb{R}^2} \mathrm{d}^2x\, \psi\Delta\dot{\psi} + \lim_{R\to\infty}\frac{R}{2\mu_0}\int_0^{2\pi} (\dot{\psi}\psi_r - \psi\dot{\psi}_r)\mathrm{d}\theta \\
&= \frac{1}{\mu_0}\int_{\mathbb{R}^2} \mathrm{d}^2x\, \psi\Delta\dot{\psi}
\end{aligned}
$$

  since $\psi, \dot{\psi} = O(r^{-1})$ and $\psi_r, \dot{\psi}_r = O(r^{-2})$.
- We need to evaluate $\Delta\dot{\psi}$

Variation of the dipolar energy

- Differentiating Poisson's equation with respect to t, we deduce that

$$\Delta\dot{\psi} = -\mu_0 M_s \vec{\nabla}\cdot\vec{\varepsilon},$$

- Hence

$$\begin{aligned}\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} E_{\mathrm{DDI}}(\vec{n}_t) &= -M_s\int_{\mathbb{R}^2}\mathrm{d}^2x\,\psi\left(\vec{\nabla}\cdot\vec{\varepsilon}\right)\\ &= M_s\int_{\mathbb{R}^2}\mathrm{d}^2x\,\vec{\varepsilon}\cdot\vec{\nabla}\psi - \lim_{R\to\infty} M_s\int_{\partial B_R(0)}\psi\star\varepsilon\\ &= M_s\int_{\mathbb{R}^2}\mathrm{d}^2x\,\vec{\varepsilon}\cdot\vec{\nabla}\psi\end{aligned}$$

by Stokes's Theorem, since $\varepsilon = \varepsilon_1\mathrm{d}x_1 + \varepsilon_2\mathrm{d}x_2$ has compact support

Evading the Hobart–Derrick Theorem

Rescaling energy and length scales

- Let us consider an energy and length rescaling with $E = E_0\hat{E}$ and $x = L_0\hat{x}$
- Then the rescaled energy is

$$\hat{E} = \hat{E}_{\mathrm{DDI}} + \int_{\mathbb{R}^3} \left\{ \frac{JL_0}{2E_0}|\mathrm{d}\vec{n}|^2 + \frac{\mathcal{D}L_0^2}{E_0}\vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n}) + \frac{K_m L_0^3}{E_0}(1 - n_z^2) + \frac{M_s B_{\mathrm{ext}} L_0^3}{E_0}(1 - n_z) \right\} \mathrm{d}^3\hat{x}$$

- For this to be dimensionless, we choose $L_0 = J/\mathcal{D}$ and $E_0 = J^2/\mathcal{D}$
- Hence, the energy can be expressed in the dimensionless form

$$\begin{aligned} \hat{E} &= \hat{E}_{\mathrm{DDI}} + \int_{\mathbb{R}^3} \left\{ \frac{1}{2}|\mathrm{d}\vec{n}|^2 + \vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n}) + \frac{K_m J}{\mathcal{D}}(1 - n_z^2) + \frac{M_s B_{\mathrm{ext}} J}{\mathcal{D}^2}(1 - n_z) \right\} \mathrm{d}^3\hat{x} \\ &\equiv \hat{E}_{\mathrm{DDI}} + \int_{\mathbb{R}^3} \left\{ \frac{1}{2}|\mathrm{d}\vec{n}|^2 + \vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n}) + K(1 - n_z^2) + h(1 - n_z) \right\} \mathrm{d}^3 x \end{aligned}$$

Rescaling energy and length scales

- Consider the rescaling of the magnetic potential $\psi = \lambda\hat{\psi}$
- The rescaled magnetostatic energy and Poisson equation are

$$\hat{E}_{\mathrm{DDI}} = \frac{1}{2}\frac{L_0\lambda^2}{\mu_0 E_0}\int_{\mathbb{R}^3}\hat{\psi}\Delta_{\hat{x}}\hat{\psi}\,\mathrm{d}^3\hat{x}, \qquad \Delta_{\hat{x}}\hat{\psi} = -\frac{\mu_0 M_s L_0}{\lambda}\vec{\nabla}_{\hat{x}}\cdot\vec{n}$$

- Introduce the dimensionless vacuum magnetic permeability

$$\mu = \frac{\mu_0 M_s L_0}{\lambda} = \left(\frac{L_0\lambda^2}{\mu_0 E_0}\right)^{-1}.$$

- Necessary magnetic potential rescaling is given by

$$\lambda = \frac{\mathcal{D}}{M_s} \quad \Longrightarrow \quad \mu = \frac{\mu_0 M_s^2 L_0}{\mathcal{D}} = \frac{\mu_0 J M_s^2}{\mathcal{D}^2}.$$

Evading the Hobart–Derrick Theorem

- The dimensionless energy and Poisson equation are

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\mathrm{d}\vec{n}|^2 + \vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n}) + K(1 - n_z^2) + h(1 - n_z) + \frac{1}{2} \vec{n} \cdot \vec{\nabla}\psi \right\} \mathrm{d}^2 x, \quad \Delta\psi = -\mu \vec{\nabla} \cdot \vec{n}$$

- Can stable topological solitons even exist in this model?
- Derrick's Theorem: If the energy functional $E[\vec{n}]$ is not stationary against spatial rescaling, then $\vec{n}$ cannot be a solution of the field equations [J. Math. Phys. **5**, 1252–1254 (1964)]
- It is a **non-existence** theorem
- Can we evade the Derrick Theorem?

Derrick's theorem applied to a linear scalar field

- Consider some arbitrary scalar field $\Phi(\vec{x})$ with associated energy

$$E[\Phi] = \int \mathrm{d}^d x \left\{ \vec{\nabla}\Phi(\vec{x}) \cdot \vec{\nabla}\Phi(\vec{x}) + V(\Phi(\vec{x})) \right\} = E_2 + E_0 \geq 0$$

- Consider coordinate rescaling $\vec{x} \mapsto \vec{x}' = \lambda \vec{x} \quad \Longrightarrow \quad \Phi_\lambda = \Phi(\lambda \vec{x})$
- Rescaled energy becomes

$$e(\lambda) = E[\Phi(\lambda\vec{x})] = \int \mathrm{d}^d x' \lambda^{-d} \left\{ \vec{\nabla}'\Phi(\vec{x}') \cdot \vec{\nabla}'\Phi(\vec{x}')\, \lambda^2 + V(\Phi(\vec{x}')) \right\} = \lambda^{2-d} E_2 + \lambda^{-d} E_0$$

- $d = 1$: $e(\lambda) = \lambda E_2 + \frac{1}{\lambda} E_0 \quad \Longrightarrow \quad e'(\lambda) = E_2 - \frac{1}{\lambda^2} E_0 = 0 \quad \Longrightarrow$ Stable topological solitons in 1D
- $d = 2$: $e(\lambda) = E_2 + \frac{1}{\lambda^2} E_0 \quad \Longrightarrow \quad e'(\lambda) = -\frac{2}{\lambda^3} E_0 = 0 \quad \Longrightarrow$ No stable topological solitons in 2D
- $d \geq 3$: $e(\lambda) = \lambda^{2-d} E_2 + \lambda^{-d} E_0 \quad \Longrightarrow \quad e'(\lambda) = (2-d)\lambda^{1-d} E_2 - d\lambda^{-(d+1)} E_0 \quad \Longrightarrow$ No stable topological solitons in $d \geq 3$

Evading the Hobart–Derrick Theorem

- The dimensionless energy and Poisson equation are

$$E = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} |\mathrm{d}\vec{n}|^2 + \vec{d}_i \cdot (\vec{n} \times \partial_i \vec{n}) + K(1 - n_z^2) + h(1 - n_z) + \frac{1}{2} \vec{n} \cdot \vec{\nabla}\psi \right\} \mathrm{d}^2 x$$

- Rescaled Poisson equation, under $\vec{x} \mapsto \vec{x}' = \lambda \vec{x}$, becomes

$$\lambda^2 \Delta' \psi_\lambda = -\mu\lambda\nabla' \cdot \vec{n}_\lambda = -\mu\lambda\nabla' \cdot \vec{n}(\lambda\vec{x}) = \lambda\Delta'\psi(\lambda\vec{x}),$$

- Magnetic potential scaling behavior is $\psi_\lambda(\vec{x}) = \frac{1}{\lambda}\psi(\lambda\vec{x})$
- Derrick scaling

$$E(\lambda) = E_{\text{exch}} + \frac{1}{\lambda} E_{\text{DMI}} + \frac{1}{\lambda^2}\left(E_{\text{pot}} + E_{\text{DDI}}\right) \quad \rightarrow \quad \left.\frac{\mathrm{d}E}{\mathrm{d}\lambda}\right|_{\lambda=1} = E_{\text{DMI}} + 2\left(E_{\text{pot}} + E_{\text{DDI}}\right) = 0$$

- Can have stable topological solitons since E_{DMI} can be negative
- Solitons can be stabilized with DMI and DDI only, potential not required

Numerical method

Numerical method

- Convenient to express everything in index notation

$$\mathcal{E} = \frac{1}{2}\left(\partial_j n_i\right)^2 + (\vec{d}_i)_j \epsilon_{jkl} n_k \partial_i n_l + K(1 - n_3^2) + h(1 - n_3) + \frac{1}{2} n_i \partial_i \psi$$

- Associated Euler–Lagrange field equations are

$$\frac{\partial \mathcal{E}}{\partial n_i} = -\partial_j \partial_j n_i + 2(\vec{d}_a)_b \epsilon_{bij} \partial_a n_j - \delta_i^3 \left(2Kn_3 + h\right) + \partial_i \psi = 0$$

- Magnetic skyrmions are are local minimizers of the energy functional

$\Rightarrow$ Solutions of the Euler–Lagrange field equations and satisfy the Derrick scaling constraint $E_{\mathrm{DMI}} + 2\left(E_{\mathrm{pot}} + E_{\mathrm{DDI}}\right) = 0$

- We choose to numerically relax the energy using an accelerated gradient descent based method with flow arresting criteria, $\partial_{tt}\vec{n} = -\mathrm{grad}\, E(\vec{n})$ [J. High Energ. Phys. **07**, 184 (2020)]

Numerical method

- Inclusion of the DDI introduces non-locality into the minimization problem
- During every iteration of the magnetization minimization, ψ must solve Poisson's equation $\Delta\psi = \mu\rho$ with source $\rho = -(\vec{\nabla}\cdot\vec{n})$
- This can be approached by reformulating the problem as an unconstrained optimization problem: minimize the functional

$$F(\psi) = \frac{1}{2}\|\mathrm{d}\psi\|_{L^2}^2 + \mu\int_{\mathbb{R}^2}\mathrm{d}^2x\left(\vec{\nabla}\cdot\vec{n}\right)\psi$$

  with respect to ψ, where the magnetization $\vec{n}$ is fixed
- We will use a non-linear conjugate gradient method with a line search strategy to solve this unconstrained problem, based on method in [J. High Energ. Phys. **06**, 116 (2024)]

Results

Isolated magnetic skyrmions

- Before implementing the numerical algorithm, we can gain some intuition by computing the divergence of the magnetization ansätze

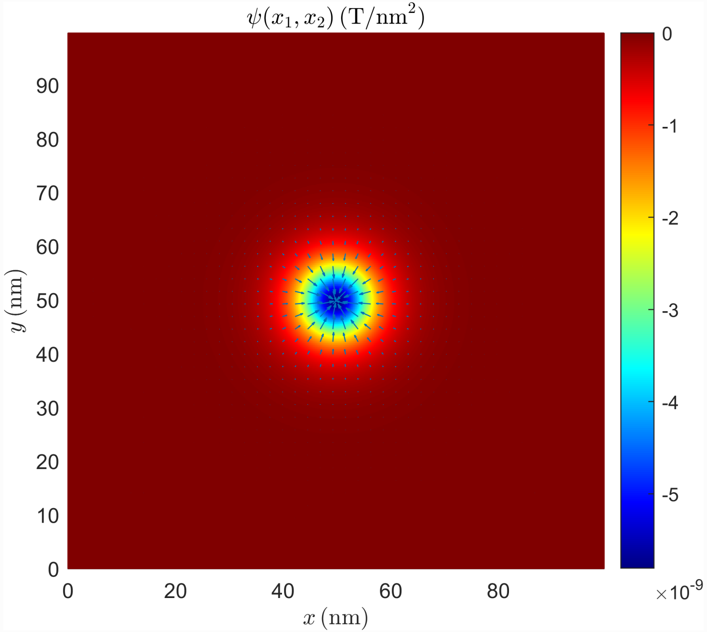
$$\vec{n}_{\text{Bloch}} = (-\sin f(r)\sin\theta, \sin f(r)\cos\theta, \cos f(r)) \implies \vec{\nabla}\cdot\vec{n}_{\text{Bloch}} = 0$$

$$\vec{n}_{\text{Néel}} = (\sin f(r)\cos\theta, \sin f(r)\sin\theta, \cos f(r)) \implies \vec{\nabla}\cdot\vec{n}_{\text{Néel}} = \frac{\mathrm{d}f}{\mathrm{d}r}\cos f(r) + \frac{1}{r}\sin f(r)$$

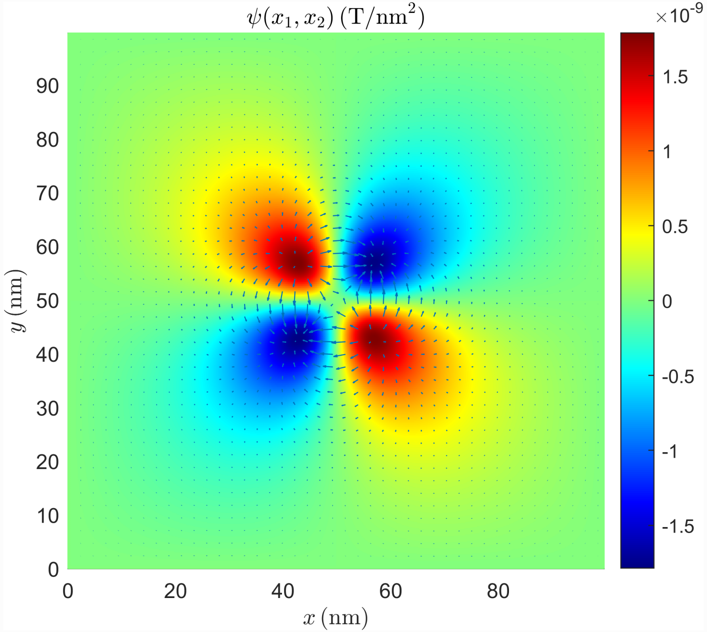
$$\vec{n}_{\text{Heusler}} = (-\sin f(r)\sin\theta, -\sin f(r)\cos\theta, \cos f(r)) \implies \vec{\nabla}\cdot\vec{n}_{\text{Heusler}} = \left(\frac{1}{r}\sin f(r) - \frac{\mathrm{d}f}{\mathrm{d}r}\cos f(r)\right)\sin 2\theta$$

- Dipolar interaction has no effect on Bloch skyrmions as the Bloch ansatz is solenoidal
- However, it does have an effect on Néel skyrmions and Heusler antiskyrmions

Magnetic potential of isolated magnetic skyrmions



(a) Rashba DMI (Néel skyrmions)

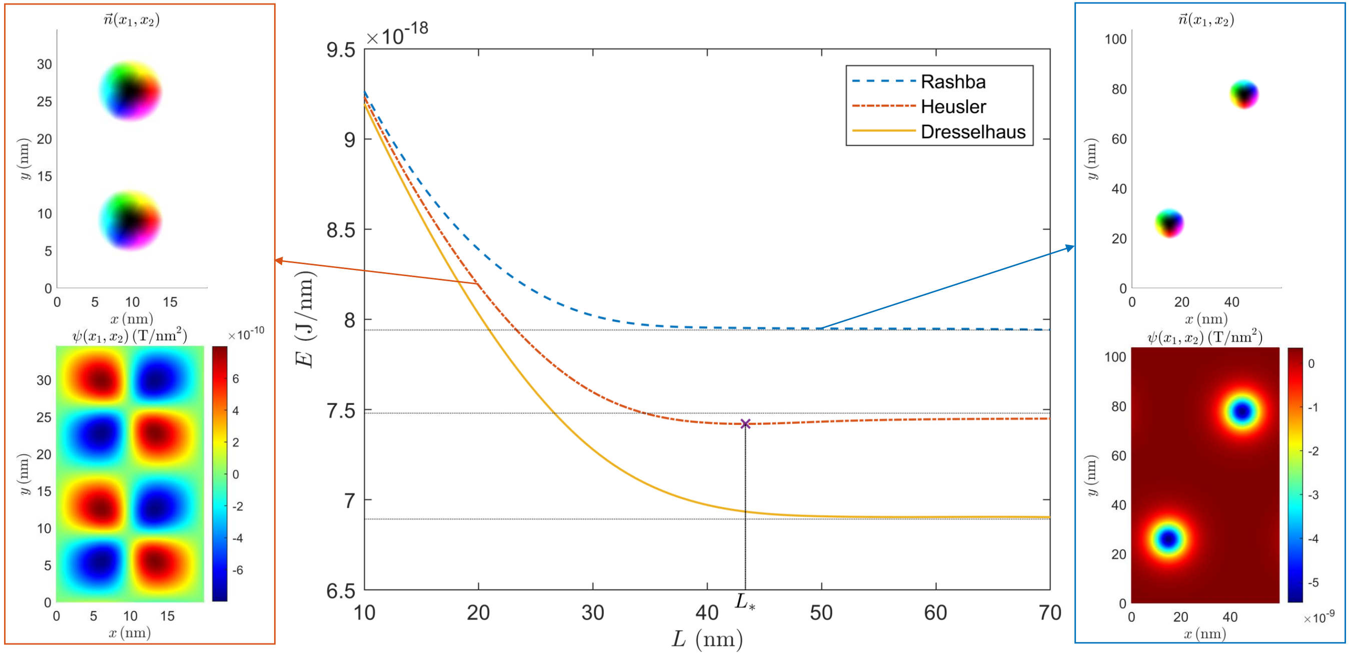


(b) Heusler DMI (antiskyrmions)

Magnetic skyrmion crystals

- Next, we investigate the DDI effect on magnetically ordered crystals
- In absence of DDI, optimal crystalline structure is hexagonal
- Restrict geometry to be equianharmonic $\implies$ rectangular unit cell of size $L \times \sqrt{3}L$
- Vary the lattice parameter L
- Initial configuration consists of two separated (anti)skyrmions
- Carried out using a product ansatz $u = u_1 + u_2$ in the $\mathbb{C}P^1$ formalism (S^2 and $\mathbb{C}P^1$ are diffeomorphic)
- In all cases, as $L \to \infty$, the energy per unit topological charge approaches that of the isolated single magnetic skyrmion

Results



$\vec{n}(x_1, x_2)$
y (nm)
x (nm)
$\psi(x_1, x_2)$ (T/nm^2)
$\times 10^{-10}$
$\times 10^{-18}$
Rashba
Heusler
Dresselhaus
E (J/nm)
L_*
L (nm)
$\times 10^{-9}$

Magnetic skyrmion crystals

- For our parameter set, interaction energy is repulsive without DDI

$\Longrightarrow$ Negative binding energy $E_{\text{bind}} = 2E_1 - E_{\text{latt}} < 0$

- Skyrmions prefer to be infinitely separated
- Remains true for Dresselhaus and Rashba DMI related skyrmions with DDI
- Heusler antiskyrmions have positive binding energy $\Longrightarrow$ finite optimal lattice size
- Heusler lattice symmetry also changes from hexagonal to square

! DDI has noticeable effect on antiskyrmions in bulk of Heusler compounds

Conclusion

Conclusion

- We have shown how to include the magnetostatic self-energy and how to compute the back-reaction
- Crystalline symmetry changed in Heusler type compounds [MRS Bull. **47**, 600 (2022)]
- Method can be extended to 3d chiral magnets (B.C.s require some care though)
- If considering thin films, stray field outside magnet needs to be determined [SIAM J. Math. Anal. **52**, 3580-3599 (2020)]
- How does the dipolar interaction effect skyrmion dynamics?
- Can our method be generalized to other systems such as skyrmions in liquid crystals? [Phys. Rev. E **90**, 042502 (2014)]

KTH

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