Compressibility of dense nuclear matter in vector meson variants of the Skyrme model

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June 24, 2024 — SIG XII, Jagiellonian University

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$\Longrightarrow$ We need to understand **phases** and **phase transitions** of nuclear matter within **vector meson variants of the Skyrme model**

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$$E(n_B)/B = E_0 + \frac{1}{2}K_0\frac{(n_B - n_0)^2}{9n_0^2} + \mathcal{O}\left((n_B - n_0)^3\right), \quad K_0 = \frac{9n_0^2}{B}\frac{\partial^2 E}{\partial n_B^2}\bigg|_{n_B=n_0}$$

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- Fudicial values: $E_0 \approx 922$ MeV and $K_0 \approx 240 \pm 20$ MeV $\Longrightarrow$ $\boxed{K_0/E_0 \approx 0.26}$

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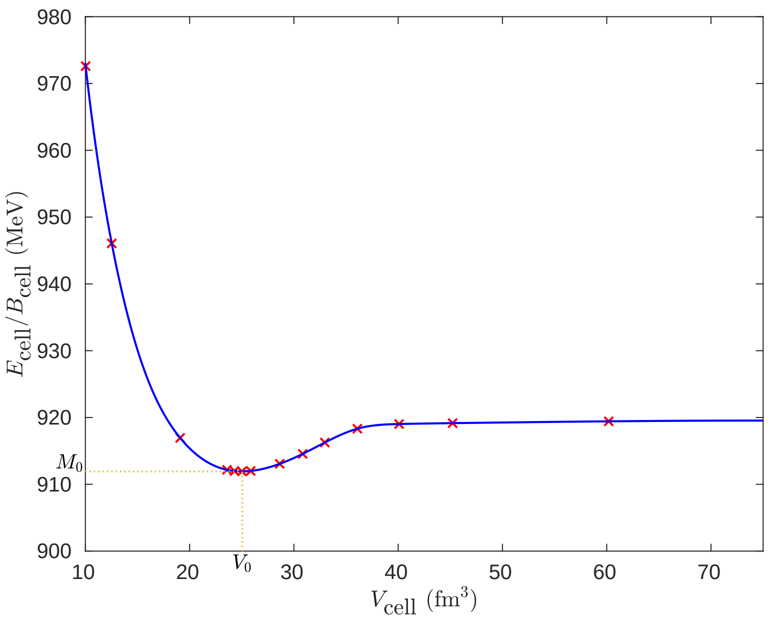
- Can consider $E_{\text{cell}}(n_B)$ by varying the baryon density n_B or, equivalently, the unit cell volume V_{cell}
- The compression modulus is thus

$$K_0 = \frac{9n_0^2}{B_{\text{cell}}} \frac{\partial^2 E_{\text{cell}}}{\partial n_B^2}\bigg|_{n_B=n_0} = \frac{9V_0^2}{B_{\text{cell}}} \frac{\partial^2 E_{\text{cell}}}{\partial V_{\text{cell}}^2}\bigg|_{V_{\text{cell}}=V_0}$$

An example: multiwall crystal [*Phys. Rev. D* **109**, 056013 (2024)]



Phases of skyrmion matter

Generalized Skyrme model

- Effective Lagrangian of mesonic fields: $\varphi : (M, g) \to (\mathrm{SU}(N_f), h)$, $N_f = 2$ (u,d-quarks)
  - Left-invariant Maurer-Cartan form $\mu \in \Omega^1(\mathrm{SU}(2)) \otimes \mathfrak{su}(2)$
  - Associated two form $\Omega \in \Omega^2(\mathrm{SU}(2)) \otimes \mathfrak{su}(2), \Omega(X, Y) = [\mu(x), \mu(Y)]$
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- Baryons realized as non-perturbative excitations of the pions $\Rightarrow$ solutions of the Euler–Lagrange field equations - topological solitons (**skyrmions**)

Skyrmion crystals

- Consider the base space to be the 3-torus

$$(\mathbb{R}^3/\Lambda, g_{\text{Euc}}), \quad \Lambda = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

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- Energy minimized over all variations of $g \iff$ optimal period lattice $\Lambda_\diamond$

Skyrmion crystals

- Given a smooth one-parameter family of variations (φ_s, g_s), skyrmion crystals are critical points of the energy $E(\varphi, g)$, that is solutions of

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- Phases of skyrmion matter $\Leftrightarrow$ fixed baryon density n_B variations of $E(\varphi, g)$
- vol_g is required to be invariant under variations g_s of the metric $\Rightarrow S_{ij} \mapsto S_{ij} - \frac{1}{\mathrm{Tr}_g(g)}S_{kl}g^{kl}g_{ij}$

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- To determine **crystalline phases**, we need to compute the **stress tensor**
- For fixed $\mathcal{L}_{024}$-field φ, there always exists a critical point of $E(\varphi, g)$ w.r.t. variations of g and it is in fact a unique c.p. (generalizes to $\mathcal{L}_{0246}$-model)
- Four crystals were found with $B_{\text{cell}} = 4$: the $\varphi_{1/2}$, φ_{α}, φ_{chain} and $\varphi_{\text{multiwall}}$ crystals

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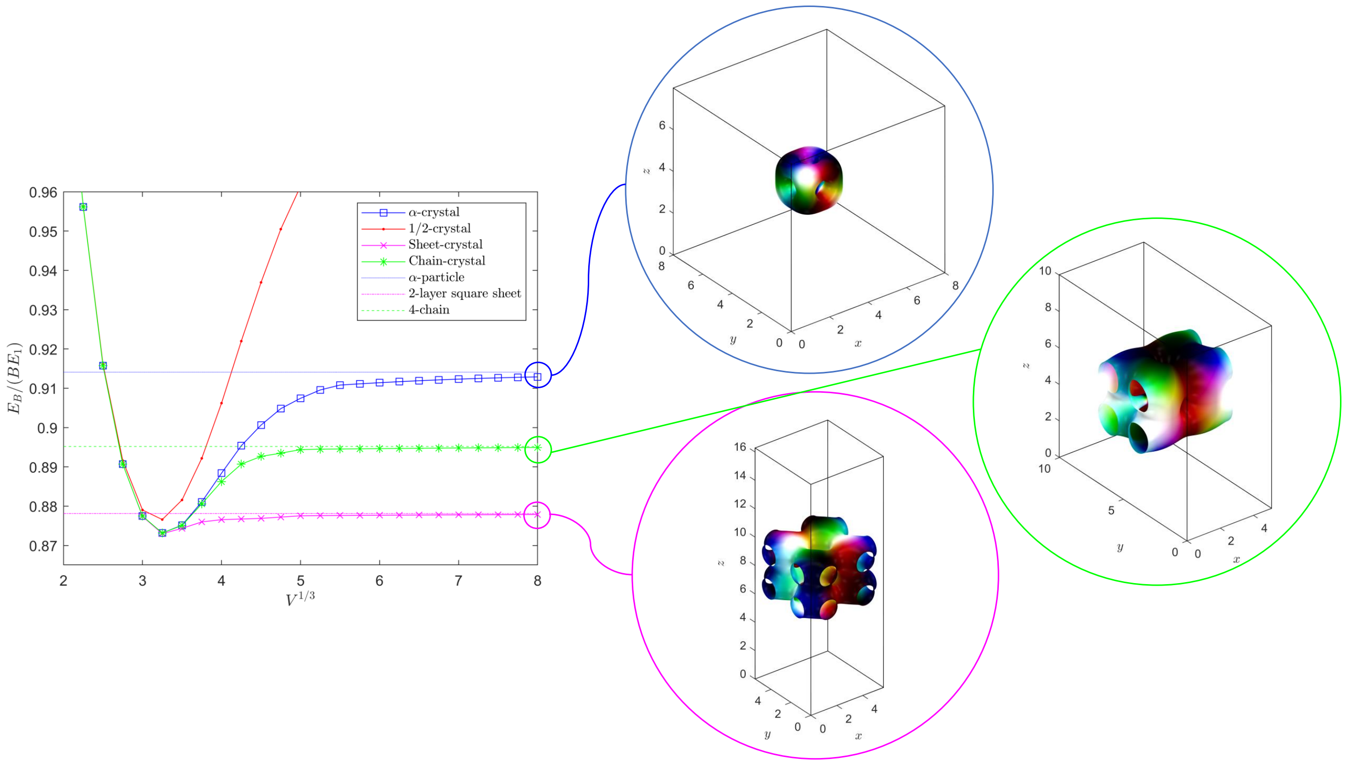
Phases of skyrmion matter



Compression modulus problem in the Skyrme model

Heuristic approach: K_0 from scaling the cubic lattice of half-skyrmions

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- Lower B.E.s are required $\implies$ inclusion of **vector mesons necessary**

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- Natural to consider replacement of *adhoc* Skyrme term by explicit interactions with finite mass vector mesons

Skyrmion crystals stabilized by ω-mesons

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- Define $\mathcal{B}_0 = *\varphi^*\Xi$ such that $\varphi^*\Xi = \mathcal{B}_0\,\mathrm{vol}_g$
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- Cauchy–Schwartz inequality:

$$B^2 = \frac{1}{c_\omega^2}\left(\int_M \omega_0\,\mathrm{vol}_g\right)^2 \leq \frac{1}{c_\omega^2}\left(\int_M \omega_0^2\,\mathrm{vol}_g\right)\left(\int_M 1\,\mathrm{vol}_g\right) = \frac{|M|}{c_\omega^2}\int_M \omega_0^2\,\mathrm{vol}_g$$

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- Yields a simple lower topological bound on the static energy,

$$E \geq \frac{1}{2}\int_M \omega^2\,\mathrm{vol}_g \geq \frac{B^2 c_\omega^2}{2|M|} \quad \xrightarrow{M=\mathbb{T}^3} \quad E \geq \frac{B^2 c_\omega^2}{2\sqrt{g}}$$

ω-Skyrme stress tensor

- The stress-energy tensor $S = S_{ij}\mathrm{d}x^i\mathrm{d}x^j$ associated to the energy

$$E(\varphi, g) = \int_M \left(\frac{1}{8}\left|\mathrm{d}\varphi\right|^2 + \frac{1}{4}(V \circ \varphi) + \frac{1}{2}\left|\mathrm{d}\omega_0\right|^2 + \frac{1}{2}\omega_0^2\right)\mathrm{vol}_g,$$

subject to the constraint

$$\left(\Delta_g + 1\right)\omega_0 = -c_\omega * \varphi^*\Xi,$$

is the section of $\mathrm{Sym}^2(T^*M)$ given by [*J. High Energ. Phys.* **06**, 116 (2024)]

$$S(\varphi, g) = \left(\frac{1}{16}|\mathrm{d}\varphi|^2 + \frac{1}{8}(V \circ \varphi) - \frac{1}{4}|\mathrm{d}\omega_0|^2 - \frac{1}{4}\omega_0^2\right)g - \left(\frac{1}{8}\varphi^*h - \frac{1}{2}\mathrm{d}\omega_0 \otimes \mathrm{d}\omega_0\right)$$

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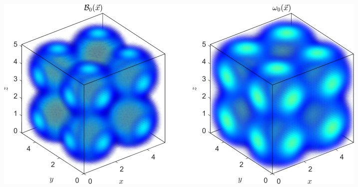
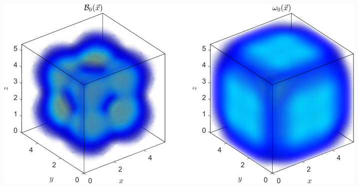
- Coincides with stress tensor of the unconstrained problem

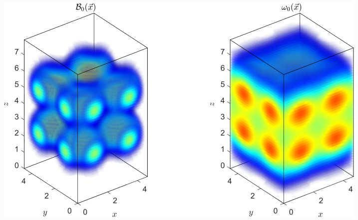
ω-skyrmion crystals



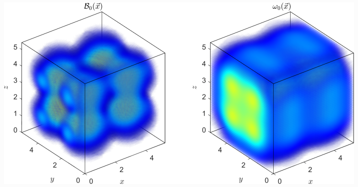
(a) $\mathrm{SC}_{1/2}$ crystal



(b) α crystal



(c) multiwall crystal



(d) chain crystal

Crystal	c_ω	E_0 (MeV)	n_0 (fm^{-3})
$\mathrm{SC}_{1/2}$	98.4	716.6	0.128
α	98.4	715.0	0.125
chain	98.4	715.0	0.125
multiwall	98.4	715.0	0.125
$\mathrm{SC}_{1/2}$	34.7	860.6	0.526
α	34.7	859.6	0.526
multiwall	34.7	859.3	0.515
chain	34.7	859.1	0.513
$\mathrm{SC}_{1/2}$	14.34	925.6	0.060
chain	14.34	922.2	0.052
α	14.34	922.1	0.051
multiwall	14.34	917.3	0.047

Bethe–Weizsäcker semi-empirical mass formula

- Can use skyrmion crystals to estimate coefficients in the Bethe–Weizsäcker SEMF

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  - $a_V \simeq 15.7 - 16.0$ MeV
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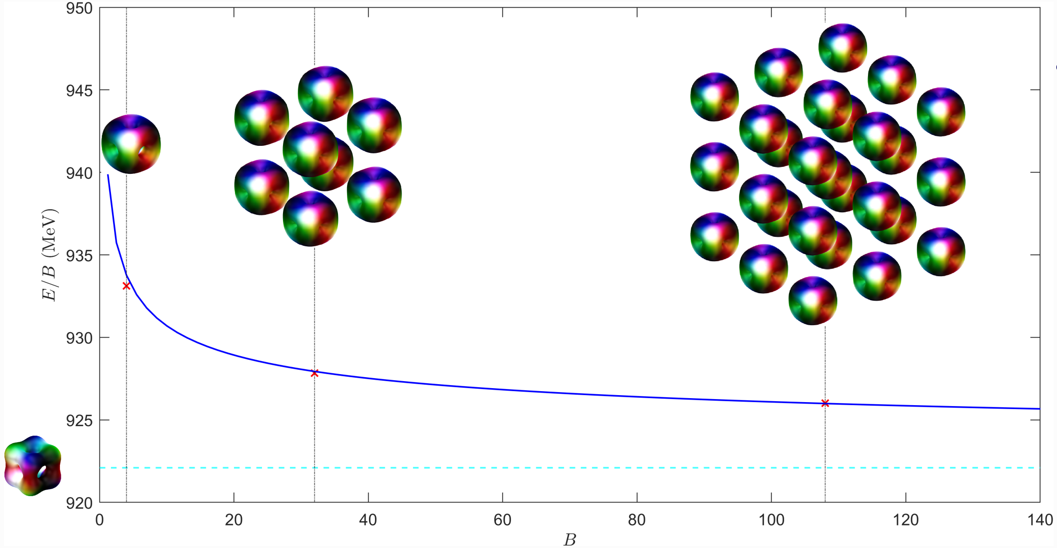
- Only need to compute the nucleon mass E_1, crystal energy $E^\alpha_{\text{crystal}}$ and the energy of a single face of an α-particle E^α_{face}



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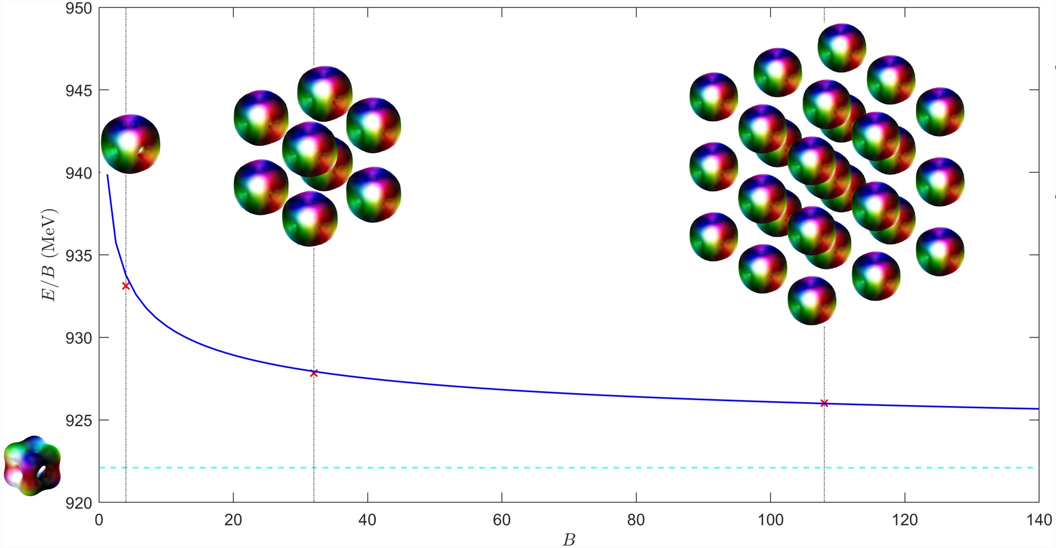

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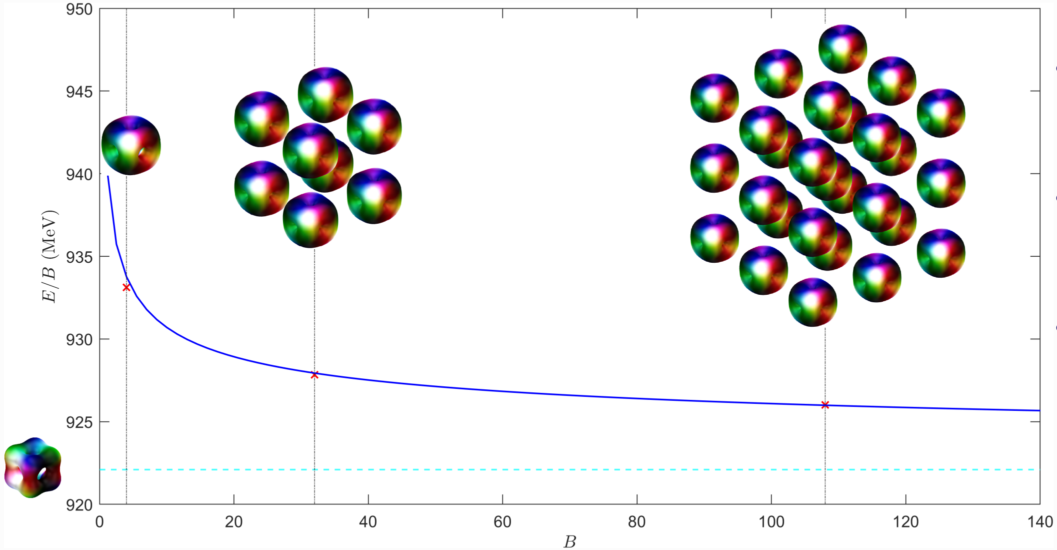

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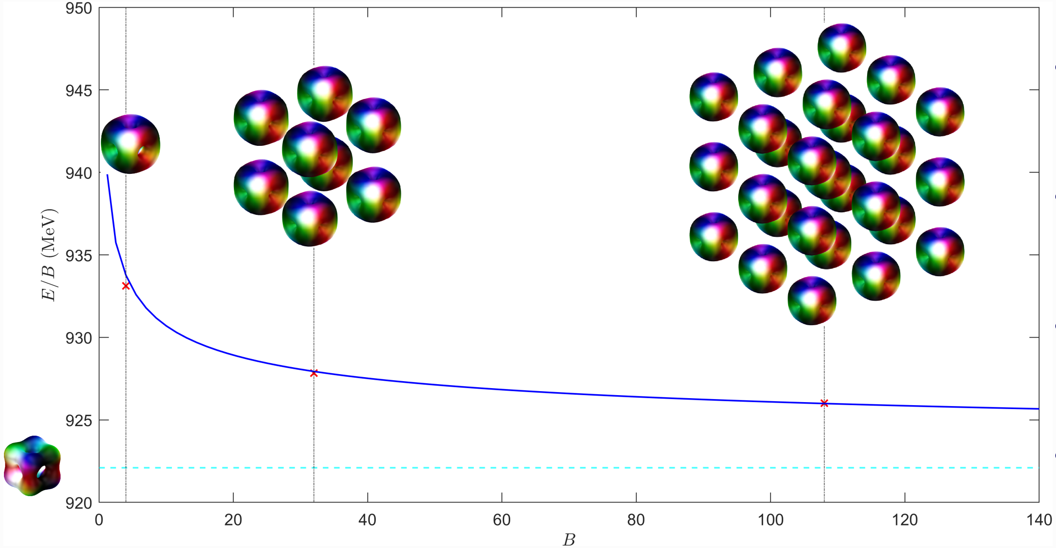

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Compressibility of ω-skyrmion Matter

- Energy of isospin symmetric nuclear matter

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Skyrmion crystals coupled to ρ-mesons

The ρ-Skyrme model

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$$\mathcal{L} = -\frac{1}{8\hbar^3}F_\pi^2 m_\pi^2 \,\mathrm{Tr}\left(\mathrm{Id} - \varphi\right) + \frac{F_\pi^2}{16\hbar}\eta^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2}\eta^{\mu\alpha}\eta^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$
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- New interaction term includes the $\rho\pi\pi$ vertex $\mathcal{L}_{\rho\pi\pi} = 2\alpha m_\rho^2 \epsilon_{abc} \rho_c^\nu \pi_a \partial_\nu \pi_b$

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- The ρ-Skyrme Lagrangian in adimensional units is

$$L = \frac{1}{2}\left\langle \mathrm{d}\varphi, \mathrm{d}\varphi \right\rangle_{L^2} - \int_M V \circ \varphi \, \mathrm{vol}_g - \frac{1}{4}\left\langle \varphi^*\Omega, \varphi^*\Omega \right\rangle_{L^2} + 8\left\{ M_\rho^2 \left\langle R, R \right\rangle_{L^2} + \frac{1}{2}\left\langle \mathrm{d}R, \mathrm{d}R \right\rangle_{L^2} + c_\alpha \left\langle \mathrm{d}R, \varphi^*\Omega \right\rangle_{L^2} \right\}$$

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- Then, the energy can be expressed as

$$\begin{aligned} E &= \frac{1}{2}\langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_{L^2} + \int_M V\circ\varphi\,\mathrm{vol}_g + \left(\frac{1}{4} - 4c_\alpha^2\right)\langle \varphi^*\Omega, \varphi^*\Omega\rangle_{L^2} + 8M_\rho^2\langle R, R\rangle_{L^2} + 4\langle \mathrm{d}R + c_\alpha\varphi^*\Omega, \mathrm{d}R + c_\alpha\varphi^*\Omega\rangle_{L^2} \\ &\geq \frac{1}{2}\langle \mathrm{d}\varphi, \mathrm{d}\varphi\rangle_{L^2} + \left(\frac{1}{4} - 4c_\alpha^2\right)\langle \varphi^*\Omega, \varphi^*\Omega\rangle_{L^2} \\ &\geq 24\pi^2\sqrt{\left(\frac{1}{4} - 4c_\alpha^2\right)}\,|B| \geq 0 \quad \Longrightarrow \quad \frac{1}{4} - 4c_\alpha^2 \geq 0 \quad \Longrightarrow \quad c_\alpha \leq \frac{1}{4} \end{aligned}$$

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- Standard hedgehog ansatz for Skyrme field

$$\varphi(r,\theta,\phi) = \left(\cos f(r), \sin f(r)\vec{n}_H(\theta,\phi)\right), \quad \vec{n}_H = \left(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta\right)$$

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- Skyrme radial profile weakly affected by ρ-meson $\rightarrow$ robustness of the hedgehog skyrmion [arXiv:2405.05731]

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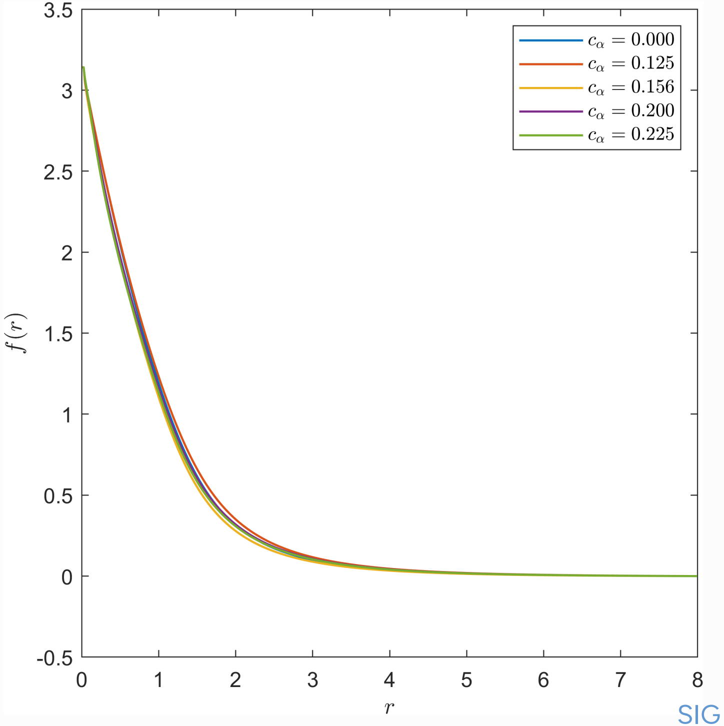

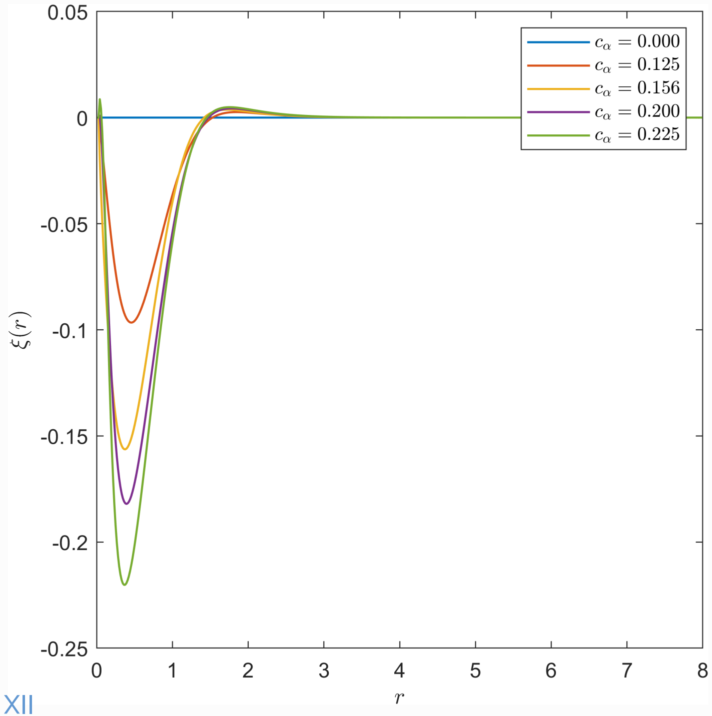

ρ-Skyrme stress tensor

- The stress-energy tensor $S = S_{ij}\mathrm{d}x^i\mathrm{d}x^j$ of $\varphi : (M, g) \to (G, h)$, associated to the energy

$$E(\varphi, \rho, g) = \int_M \left(\frac{1}{2}|\mathrm{d}\varphi|^2 + \frac{1}{4}|\varphi^*\Omega|^2 + (V \circ \varphi) + 8M_\rho^2|R|^2 + 4|\mathrm{d}R|^2 + 8c_\alpha|\langle \mathrm{d}R, \varphi^*\Omega\rangle|^2\right)\mathrm{vol}_g,$$

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- Proof is a simple extension of the standard $\mathcal{L}_{024}$-Skyrme model proof

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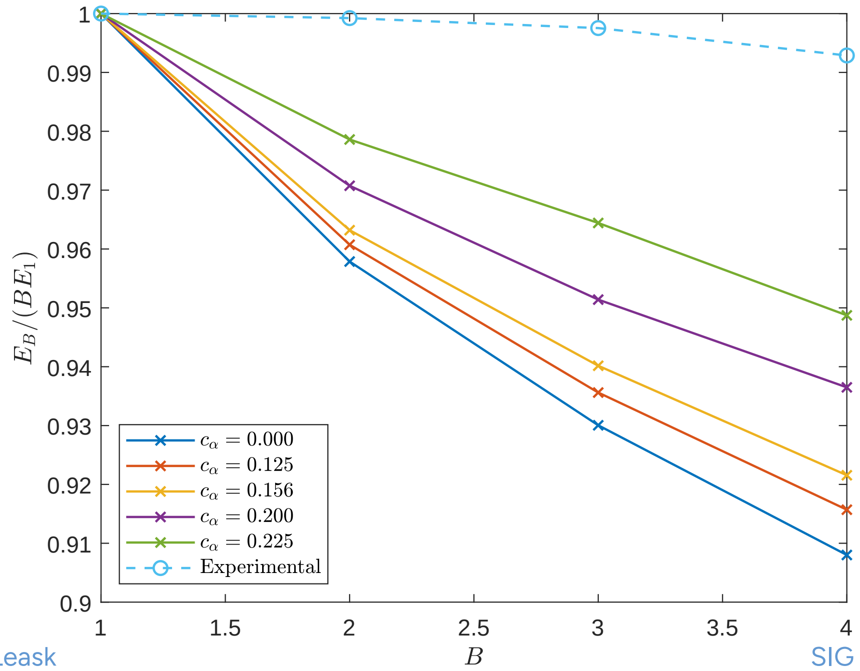
- Here we have used the identities $\mathrm{Tr}_g\,\varphi^*h = |\mathrm{d}\varphi|^2$ and $\mathrm{Tr}_g\,A\cdot B = -2|\langle A,B\rangle|^2$

Multi-skyrmions with ρ-mesons



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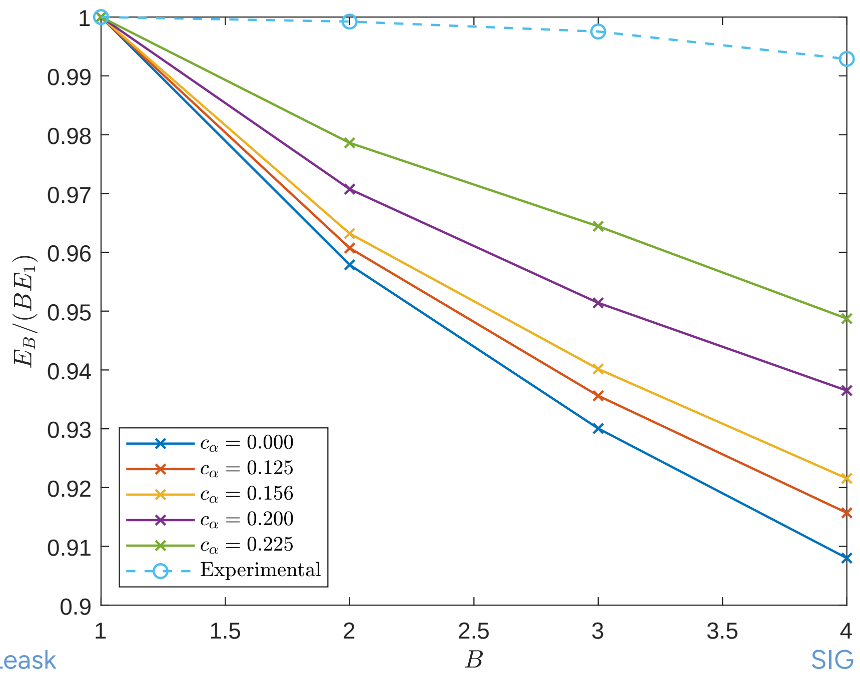

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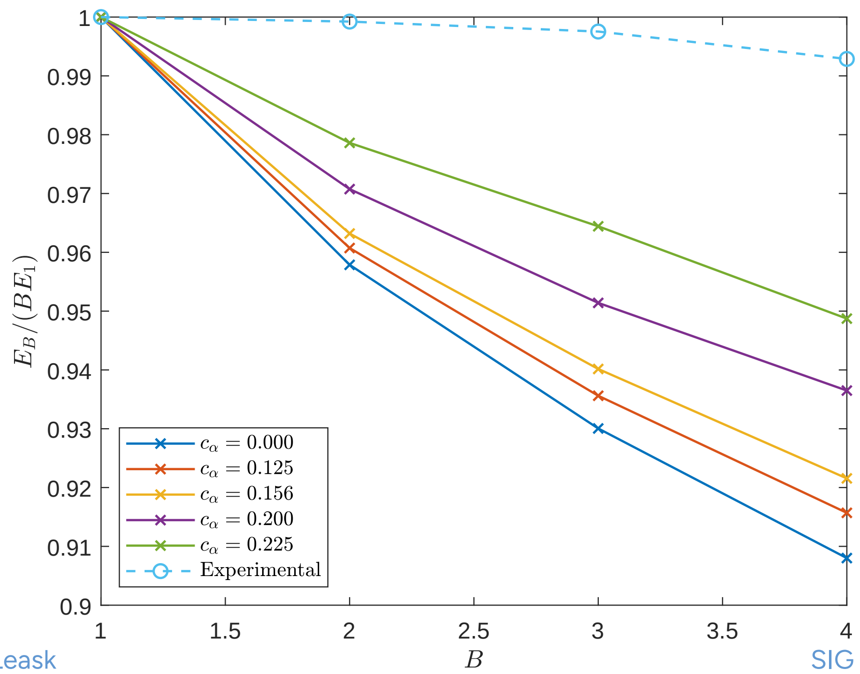

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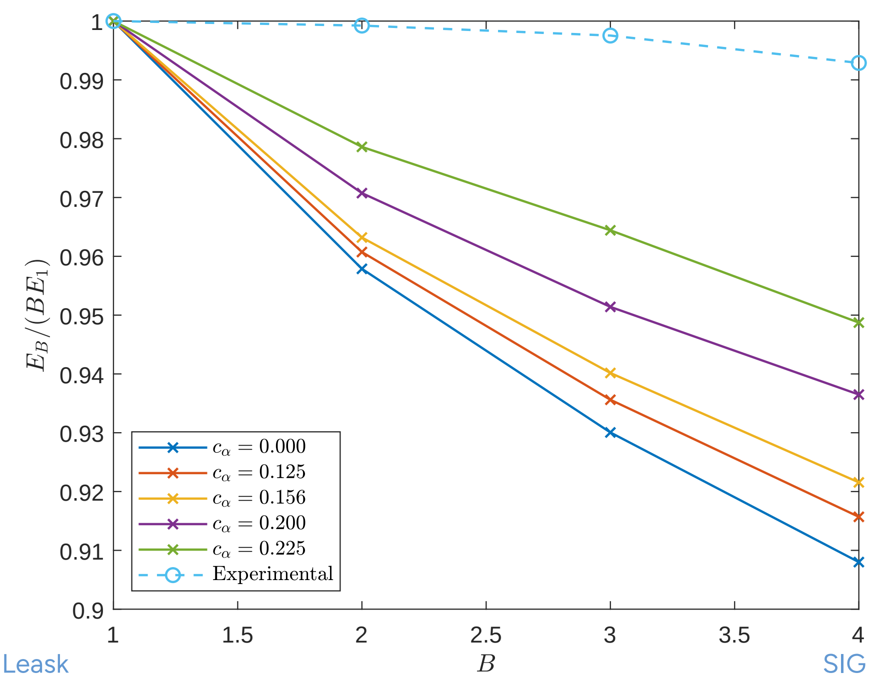

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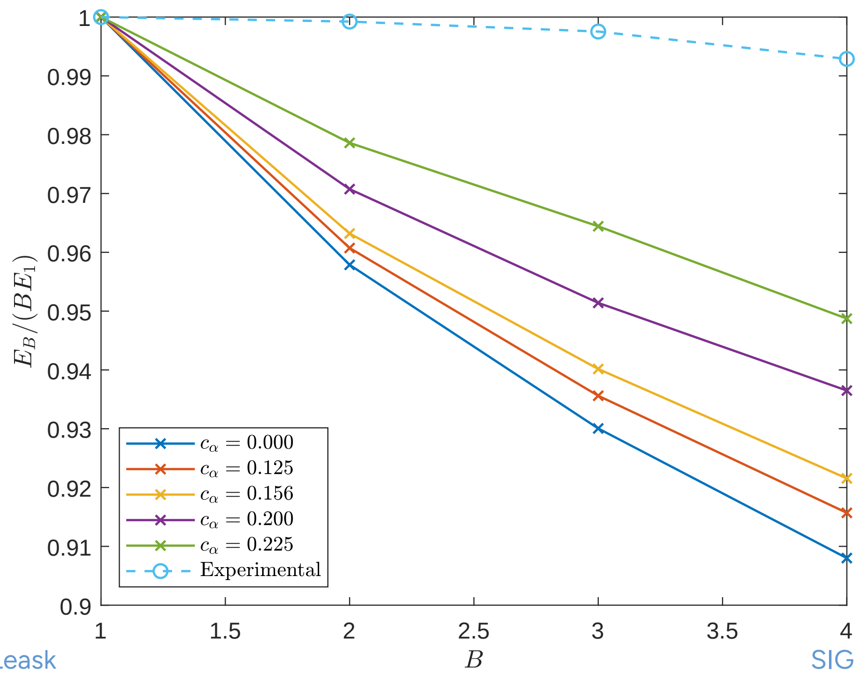

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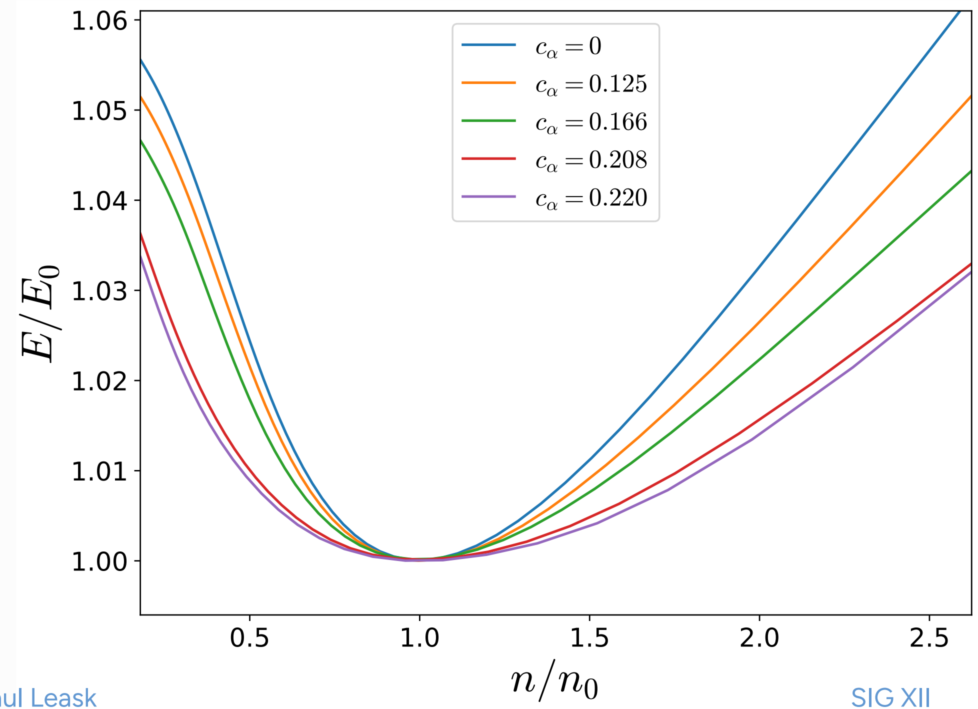
$\Longrightarrow$ Lower compression modulus K_0

Compressibility of ρ-skyrmion Matter



- Ground state crystal found to be the α-particle crystal for all c_α
- K_0 decreases with binding energies

c_α	K_0/E_0	K_0(MeV)	B.E. (%)
0	1.170	1080	5.54
0.125	0.985	909	5.36
0.166	0.778	718	5.00
0.208	0.461	425	4.25
0.220	0.381	351	3.85

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  - Numerically time consuming, can we find approximations to the crystal solutions in the vector meson theories?
    - Half crystal approximation for skyrmions coupled to vector mesons [*Nucl. Phys. A* **736**, 129–145 (2004)]
    - Approximate multiwall crystals? [*J. Phys. A: Math. Theor.* **42**, 482001 (2009)]
    - Inhomogeneous planar crystals from instantons? [*Nucl. Phys. A* **989**, 231–245 (2019)]