Neutron stars from skyrmion branes

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- Can these neutron stars have sufficient maximal masses?

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- In the large N_c-limit, QCD can be reduced to an effective field theory of mesons
- Skyrme's idea [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]: effective mesonic Lagrangian involving only **pions**, with **baryons emerging as stable topological solitons**

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$$\mathrm{SU}(2) \ni \begin{pmatrix} \sigma + i\pi_3 & i\pi_1 + \pi_2 \\ i\pi^1 - \pi_2 & \sigma - i\pi_3 \end{pmatrix} \leftrightarrow (\sigma, \pi_1, \pi_2, \pi_3) \in S^3$$

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- Disjoint homotopy classes labelled by $B \in \pi_3(S^3) = \mathbb{Z}$

$\Longrightarrow$ Fields are **topologically stable** and B is identified with the **baryon number**

Generalized Skyrme Lagrangian

- Skyrme's original model:

$$\mathscr{L}_{24} = \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$

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- This is $(\mathrm{SU}(2) \times \mathrm{SU}(2))/\mathbb{Z}_2 \cong \mathrm{SO}(4)$ invariant and the **pions are massless**
- Boundary condition $\varphi(\vec{x} \to \infty) = \mathbb{I}_2$ spontaneously breaks chiral $\mathrm{SO}(4)$ symmetry to an isospin $\mathrm{SO}(3)$ symmetry, which acts on $\vec{\pi}$

Generalized Skyrme Lagrangian

- The standard massive Skyrme model includes the pion mass potential:

$$\mathscr{L}_{024} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\mathrm{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar} g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2} g^{\mu\alpha} g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$

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$$\mathcal{L}_{0246} = -\frac{F_\pi^2 m_\pi^2}{8\hbar^3}\mathrm{Tr}\left(\mathbb{I}_2 - \varphi\right) + \frac{F_\pi^2}{16\hbar}g^{\mu\nu}\,\mathrm{Tr}(L_\mu L_\nu) + \frac{\hbar}{32e^2}g^{\mu\alpha}g^{\nu\beta}\,\mathrm{Tr}\left([L_\mu, L_\nu][L_\alpha, L_\beta]\right)$$
$$- \pi^4\lambda^2 g^{\mu\nu}\mathcal{B}_\mu\mathcal{B}_\nu, \qquad \lambda^2 = g_\omega^2/(2\pi^4 m_\omega^2)$$

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- Exhibits short range ω-meson-like repulsion while still describing scalar meson effects
- Baryon d.o.f. not explicitly visible $\longrightarrow$ topology: Homotopy invariant $\leftrightarrow$ Baryon number

$$\pi_3(\mathrm{SU}(2)) = \mathbb{Z} \ni B = \int_{\mathbb{R}^3} \mathrm{d}^3x\sqrt{-g}\,\mathcal{B}^0, \quad \mathcal{B}^\mu = \frac{1}{24\pi^2\sqrt{-g}}\epsilon^{\mu\nu\rho\sigma}\operatorname{Tr}(L_\nu L_\rho L_\sigma)$$

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- Baryons realized as non-perturbative excitations of the pions $\Longrightarrow$ solutions of the Euler–Lagrange field equations - topological solitons (**skyrmions**)

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- We are interested in **static** solutions and adopt the usual Skyrme units of length $\tilde{L} = 2\hbar/eF_\pi$ and energy $\tilde{E} = F_\pi/4e$

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$$\frac{\pi^4\lambda^2 e^4 F_\pi^2}{2\hbar^3} = c_6$$

$$\begin{aligned} T_{\mu\nu} &= -\frac{2}{\sqrt{-g}}\frac{\partial\left(\sqrt{-g}\mathscr{L}_{0246}\right)}{\partial g^{\mu\nu}} \\ &= -\operatorname{Tr}(L_\mu L_\nu) - \frac{1}{4}g^{\alpha\beta}\operatorname{Tr}([L_\mu, L_\alpha][L_\nu, L_\beta]) + 2c_6\mathscr{B}_\mu\mathscr{B}_\nu + g_{\mu\nu}\mathscr{L}_{0246} \end{aligned}$$

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- The adimensional static energy is thus ($T_{00} = \mathcal{E}_{\text{stat}} + \mathcal{E}_{\text{kin}}$)

$$M_B(\varphi, g) = \int_{\mathbb{R}^3} \mathrm{d}^3x\sqrt{-g}\,\mathcal{E}_{\text{stat}}$$
$$= \int_M \mathrm{d}^3x\sqrt{-g}\left\{-\frac{1}{2}g^{ij}\operatorname{Tr}(L_iL_j) - \frac{1}{16}g^{ik}g^{jl}\operatorname{Tr}\left([L_i, L_j][L_k, L_l]\right)\right.$$

$$m = \frac{2m_\pi}{F_\pi e} \rightarrow \qquad \left. +m^2\operatorname{Tr}\left(\mathbb{I}_2 - \varphi\right) + c_6\frac{\epsilon^{ijk}\epsilon^{abc}}{(24\pi^2\sqrt{-g})^2}\operatorname{Tr}(L_iL_jL_k)\operatorname{Tr}(L_aL_bL_c)\right\}$$

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- Skyrmions are **energy minimizing** static solutions of the Euler–Lagrange equations associated to M_B

Linking in the Skyrme model

Baryon number as the linking number of vortices

- The Skyrme field can be written in terms of two vortices $\psi_1, \psi_2 \in \mathbb{C}$ as

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- The map $\varphi : \mathbb{R}^3 \cup \{\infty\} \cong S^3 \to S^3$ of degree B has Hopf charge $Q = B$ under the Hopf map $H : S^3 \to S^2$ [*Phys. Rev. D* **101**, 065011 (2020)]

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- Distinct regular points on S^2 under $H \circ \varphi : \mathbb{R}^3 \cup \{\infty\} \to S^2$ have preimages on $\mathbb{R}^3 \cup \{\infty\}$ that are linked $Q = B$ times
- Antipodal points on S^2 identified as **vortex lines** (zeros) and are linked $Q = B$ times

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Skyrmion solutions

$B = 1$ skyrmion [*Proc. R. Soc. Lond. A* **260** 127-138 (1961)]

- The hedgehog field is $\varphi(\vec{x}) = \exp\left(i f(r) \vec{x} \cdot \vec{\tau}\right)$

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- The hedgehog solution has baryon number $B = 1$ since

$$B = -\frac{1}{2\pi^2}\int_0^\infty \frac{\sin^2 f}{r^2}\frac{\mathrm{d}f}{\mathrm{d}r}4\pi r^2 dr = \frac{1}{\pi}f(0) = 1$$

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- E–L equations reduce to a 2nd order non-linear ODE that can only be solved numerically,

$$\left(r^2 + 2\sin^2 f\right)\frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}f}{\mathrm{d}r} + \sin 2f\left[\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 - 1 - \frac{\sin^2 f}{r^2}\right] = 0$$

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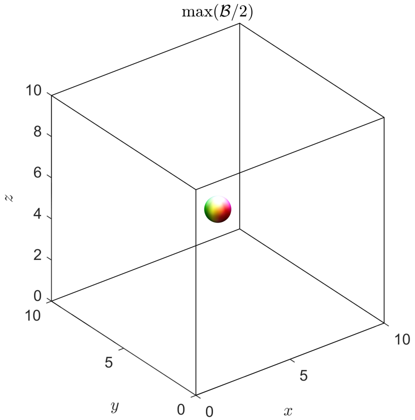

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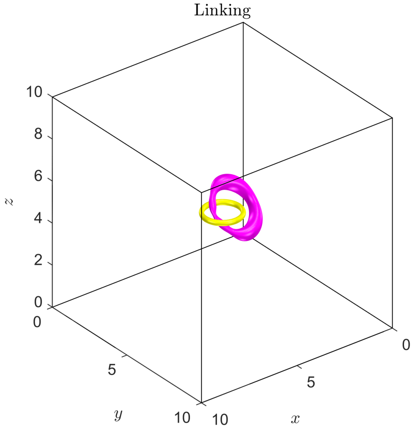

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- Rational maps for $B = 1, \ldots, 4$:

B	$R(z)$	Symmetry
1	z	$O(3)$
2	z^2	$O(2)\times\mathbb{Z}$
3	$\frac{z^3-\sqrt{3}iz}{\sqrt{3}iz^2-1}$	T_d
4	$\frac{z^4+2\sqrt{3}iz^2+1}{z^4-2\sqrt{3}iz^2+1}$	O_h

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- Substituting the RMA into the massless static energy functional yields

$$M_B = 4\pi \int_0^\infty r^2 \left\{ \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 2B \sin^2 f \left(\left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 1 \right) + \mathcal{I} \frac{\sin^4 f}{r^2} \right\} \mathrm{d}r,$$

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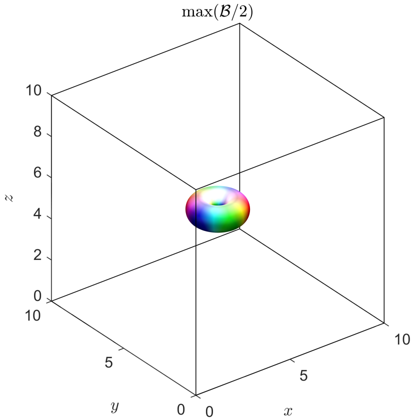
- Optimising $\mathcal{I}$ and the profile function $f(r)$ gives approximate Skyrmions, but further numerical relaxation is required to find true Skyrmions.

$B = 2$ skyrmion [*Phys. Lett. B* **195**, 235–239 (1987)]



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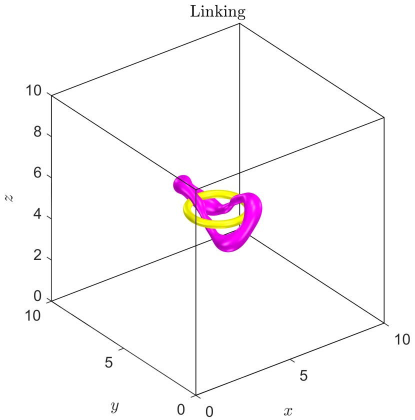

(b) Linking of two preimages

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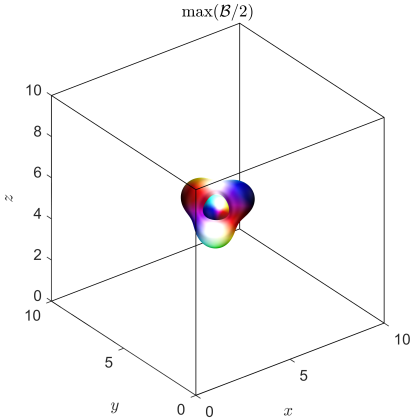

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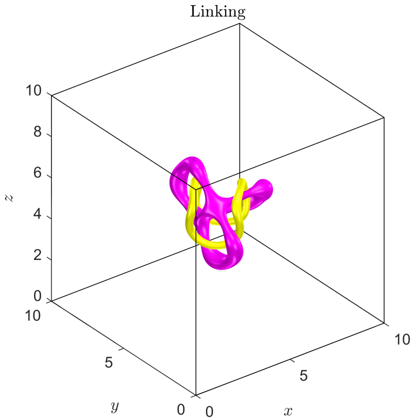

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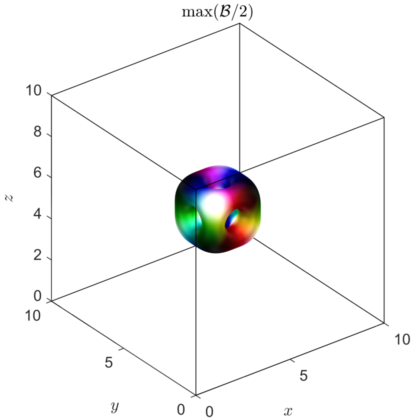

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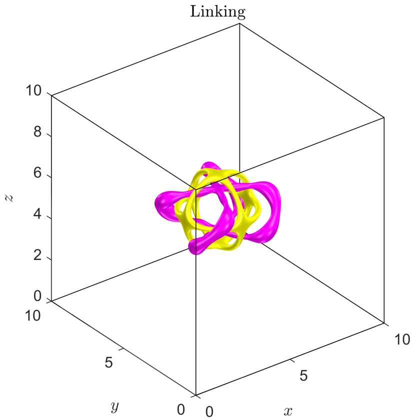

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Skyrmion crystals and phases of skyrmion matter

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- Ground state of dense nuclear matter has a **crystalline** structure in the classical approximation
- In order to determine skyrmion crystals, we first need to define what a crystal really is!

Skyrmion crystals

- Skyrme crystals are energy minimizing maps

$$\varphi : \mathbb{R}^3/\Lambda_\diamond \to SU(2), \quad \Lambda_\diamond = \left\{ n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z} \right\}$$

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- Key idea [*Comm. Math. Phys.* **332** 355-377 (2014)]: Identify all 3-tori via diffeomorphism (with $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$)

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, d), \quad (x^1, x^2, x^3) \mapsto x^1\vec{X}_1 + x^2\vec{X}_2 + x^3\vec{X}_3$$

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- Energy minimized over variations of $g \iff$ energy minimizing period lattice $\Lambda_\circ$

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- For fixed $\mathscr{L}_{024}$-field φ, there always **exists** a critical point of $M_B(\varphi, g)$ w.r.t. variations of g and it is in fact a **unique** c.p. (generalizes to $\mathscr{L}_{0246}$-model)

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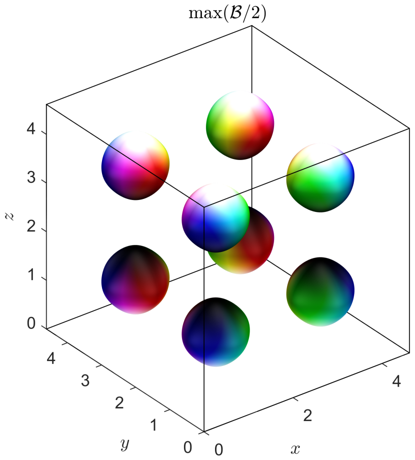
$$\sigma = -c_1 c_2 c_3, \quad \pi_1 = s_1 \sqrt{1 - \frac{s_2^2}{2} - \frac{s_3^2}{2} + \frac{s_2^2 s_3^2}{3}}, \quad \text{and cyclic,}$$

where $s_i = \sin(2\pi x^i / L)$ and $c_i = \cos(2\pi x^i / L)$, with initial metric $g = L^2 \mathbb{I}_3$.

Half-skyrmion (FCC) crystal [*Phys. Lett. B* **208** 491–494 (1988)]



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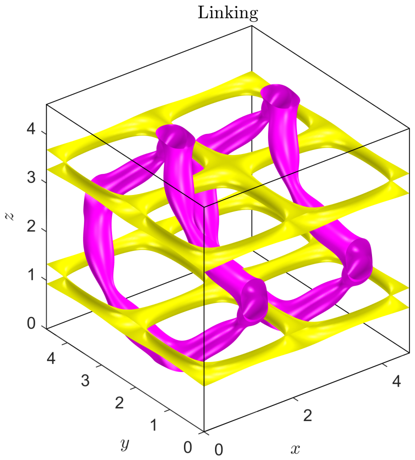

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$\implies$ Should yield a **lower compression modulus** than previous studies

Summary of [*J. Math. Phys.* **64** 103503 (2023)]

- From $\varphi_{\rm FCC}$, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{\rm FCC}$, and minimizing M_B w.r.t. variations of φ and g
- These are

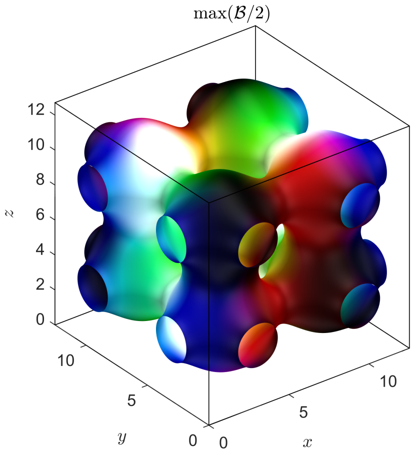
$$Q \in \left\{ \mathbb{I}_4, \underbrace{\begin{pmatrix} (0,-1,1,1)/\sqrt{3} \\ * \end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix} (0,0,0,1) \\ * \end{pmatrix}}_{Q_{\rm brane}}, \underbrace{\begin{pmatrix} (0,0,1,1)/\sqrt{2} \\ * \end{pmatrix}}_{Q_{\rm string}} \right\}.$$

- The $\varphi_{\rm brane}$-crystal is the lowest energy solution at all baryon densities $n_B = B_{\rm cell}/V_{\rm cell}$
- $\Delta E = E_{\rm isolated} - E_{\rm min}$ is minimized for the choice of crystal $\varphi_{\rm brane}$

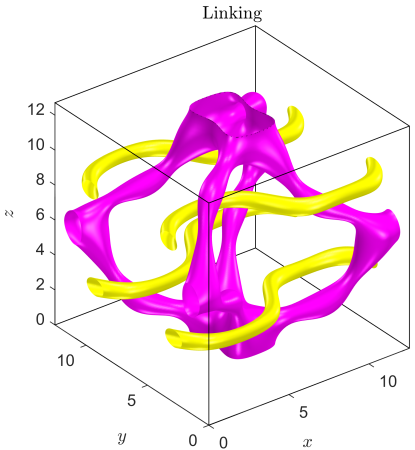
$\Longrightarrow$ Should yield a **lower compression modulus** than previous studies

$\Longrightarrow$ **Brane crystal** is an ideal candidate for **dense nuclear matter**

Brane or domain wall crystal [*J. Math. Phys.* **64** 103503 (2023)]



(a) Isosurface plot of the baryon density $\mathscr{B}_0$



(b) Linking of two preimages

Varying the metric on $\mathbb{T}^3$

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- $S(\varphi, g)$ is the **stress-energy tensor**:

$$S_{ij} = \frac{1}{2}\left[m^2\, \mathrm{Tr}(\mathrm{Id} - \varphi) - \frac{1}{2} g^{kl}\, \mathrm{Tr}(L_k L_l) - \frac{1}{16} g^{km} g^{ln}\, \mathrm{Tr}([L_k, L_l][L_m, L_n]) - c_6 (B_0)^2 \right] g_{ij}$$
$$+ \frac{1}{2} \mathrm{Tr}(L_i L_j) + \frac{1}{8} g^{kl}\, \mathrm{Tr}([L_i, L_k][L_j, L_l]).$$

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$

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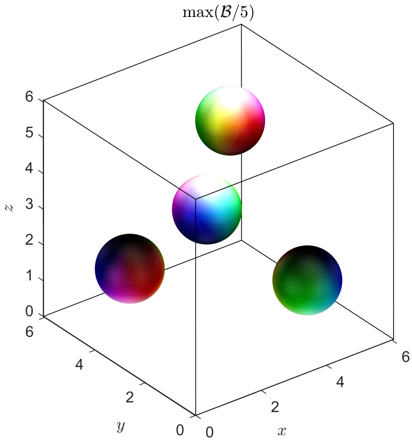
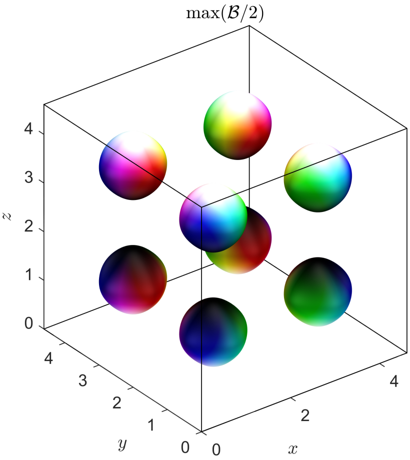
$\Longrightarrow$ Laddering of minimizations

An example: the FCC to half-skyrmion crystal



(a) Initial configuration of four $B = 1$ hedgehogs arranged in the attractive channel on a cubic grid



(b) Relaxed final solution of the cubic crystal of half-skyrmions

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- This is key to obtaining an **equation of state** within our framework

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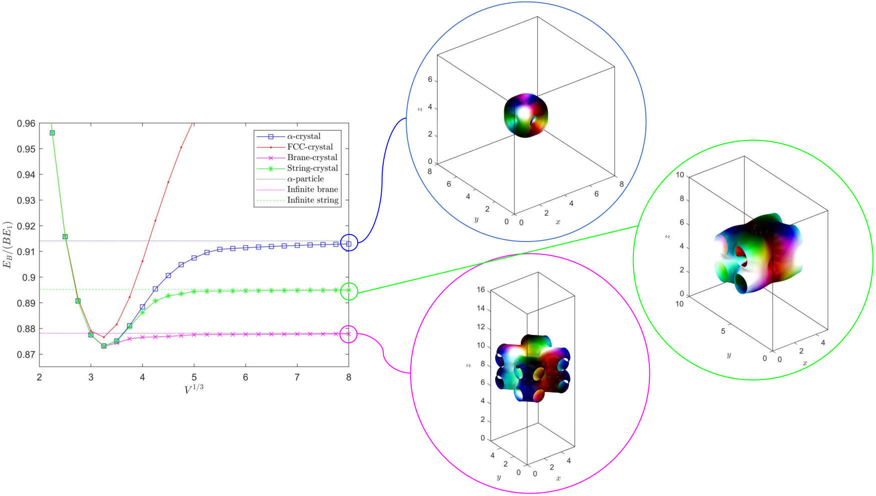

Quantum skyrmion crystals and the symmetry energy

Isospin quantization

- Non-renormalizable theory $\Longrightarrow$ isospin asymmetry is included by semi-classically quantizing isospin collective coordinates: $\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^{\dagger}(t)$ [*Nucl. Phys. B* **262** 133–143 (1985)]

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- The isospin correction (per unit cell) to the energy of the crystal is found to be $E_{\text{iso}} = \frac{\hbar^2}{8U_{33}}B_{\text{cell}}^2\delta^2$

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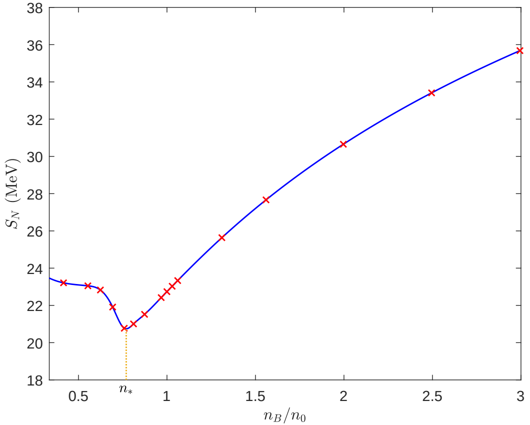
- In [arXiv:2306.04533 (2023)], at saturation we find

$$\boxed{n_0 = 0.160\,\mathrm{fm}^{-3},\ E_N(n_0) = 912\,\mathrm{MeV} \text{ and } S_N(n_0) = 22.7\,\mathrm{MeV}}$$

Symmetry energy and the cusp structure



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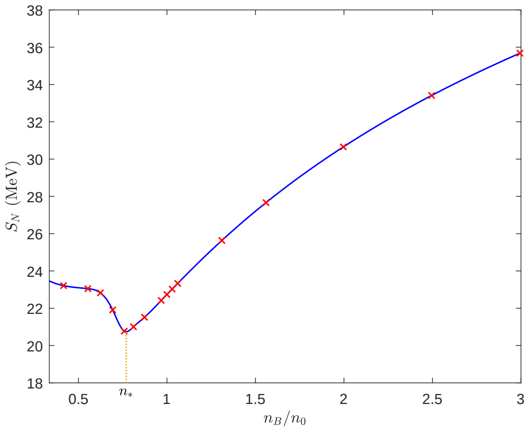

- **Cusp** below saturation at $n_* \sim 3n_0/4$

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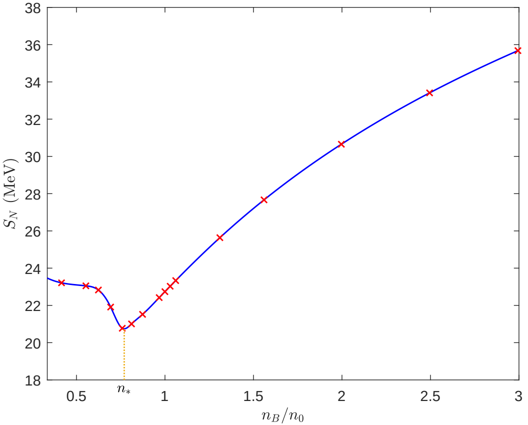

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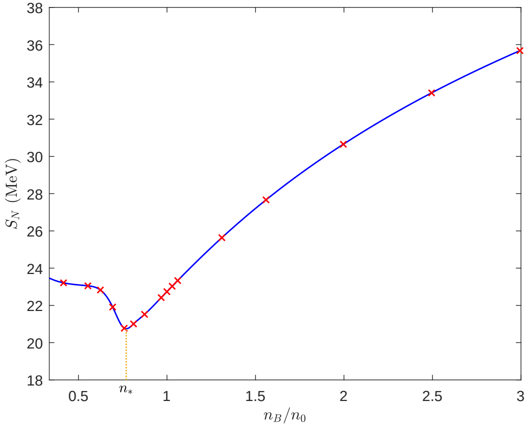

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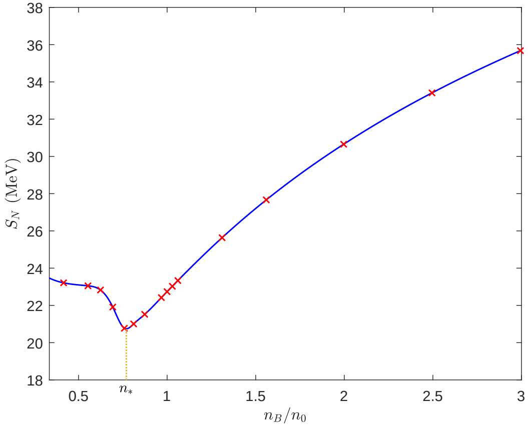

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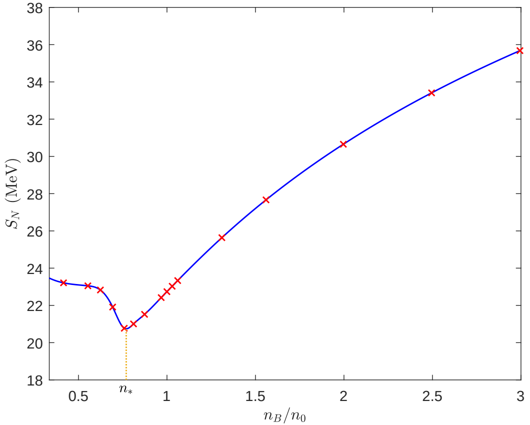

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- Cusp origin: **phase transition** between **infinite isospin asymmetric nuclear matter** and somewhat **isolated finite nuclear matter** [arXiv:2306.04533 (2023)]

Particle fractions of $npe\mu$ matter in β-equilibrium

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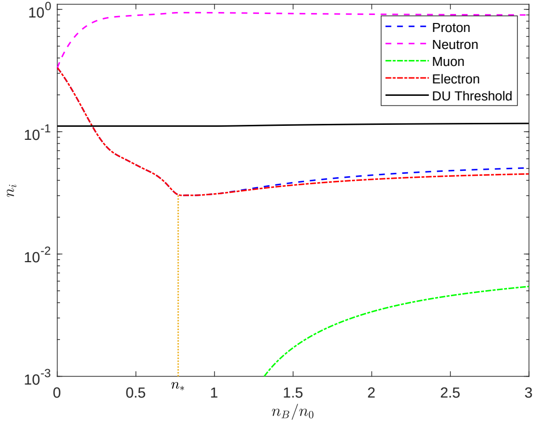
Particle fractions of $npe\mu$ matter in β-equilibrium



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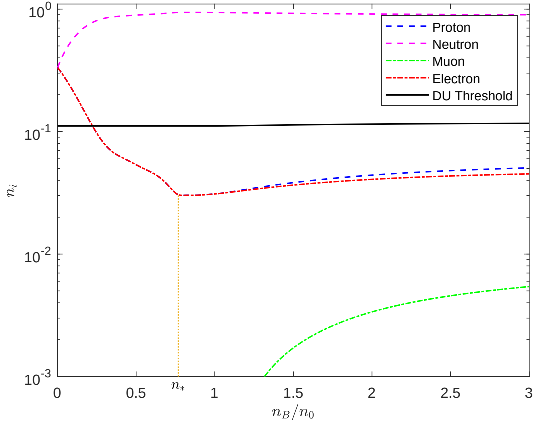

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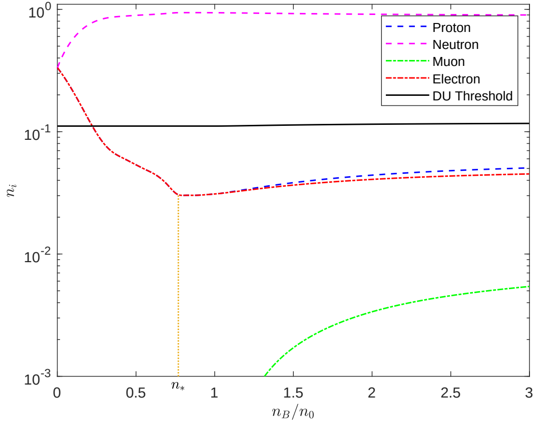

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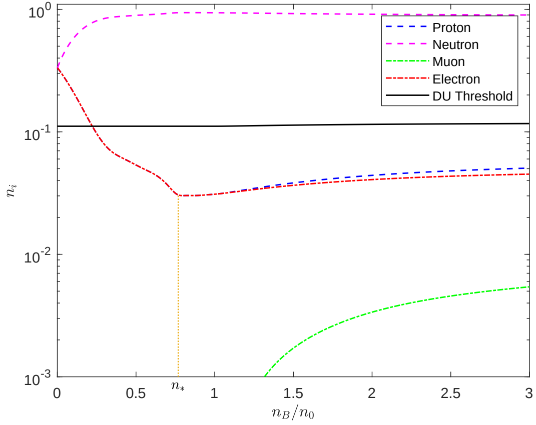

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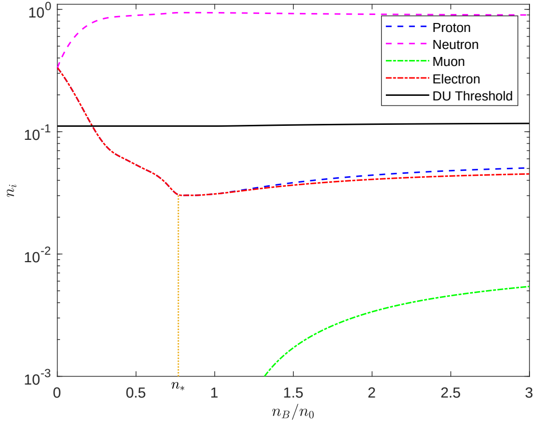

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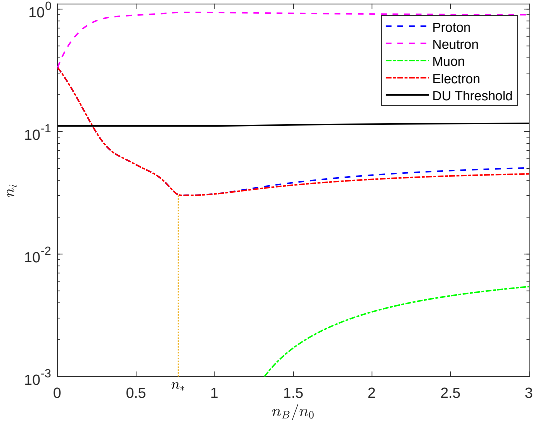

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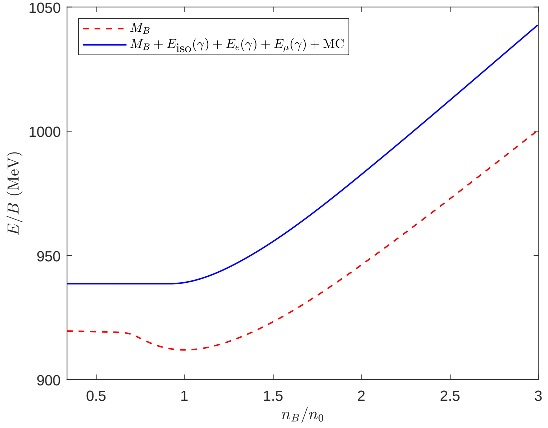
Isospin asymmetric equation of state



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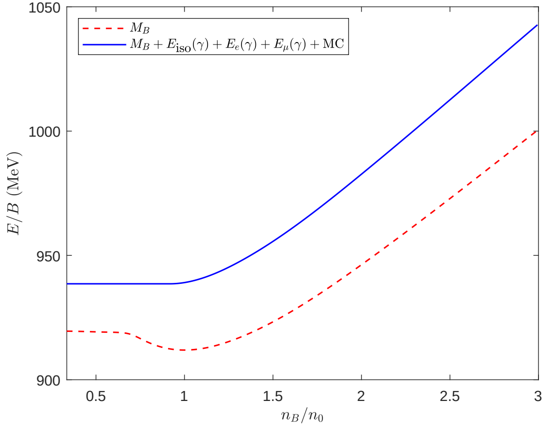

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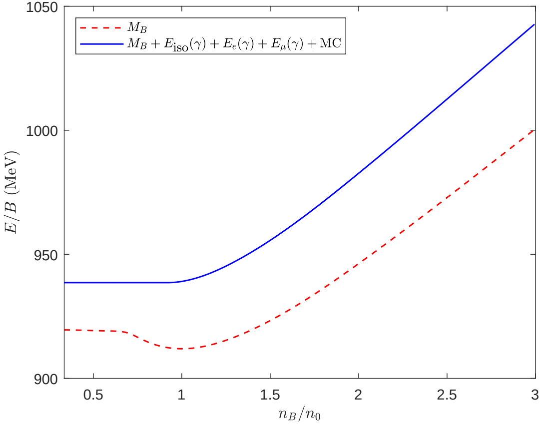

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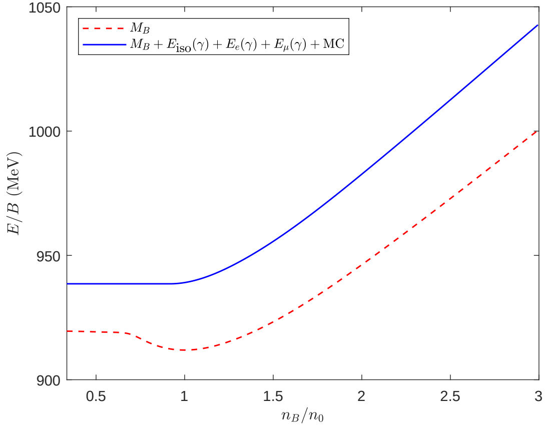

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- We will use this EoS to obtain NS within the Skyrme model

Neutron stars

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- The energy density ρ and the pressure p are related by the (Brane) crystal EoS $\rho(p) = \rho_{\text{brane}}(p)$ [*Phys. Lett. B* **811** 135928 (2020)]

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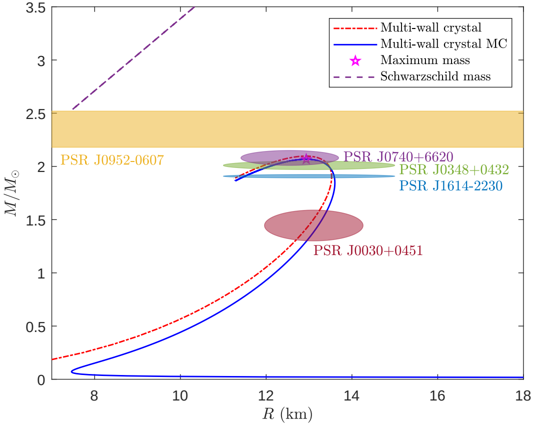
$$\frac{\mathrm{d}A}{\mathrm{d}r} = A(r)r\left(8\pi G B(r)p(r) - \frac{1-B(r)}{r^2}\right)$$

$$\frac{\mathrm{d}B}{\mathrm{d}r} = B(r)r\left(8\pi G B(r)\rho(p(r)) + \frac{1-B(r)}{r^2}\right)$$

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{p(r)+\rho(p(r))}{2A(r)}\frac{\mathrm{d}A}{\mathrm{d}r}$$

Neutron star properties and the mass-radius curve



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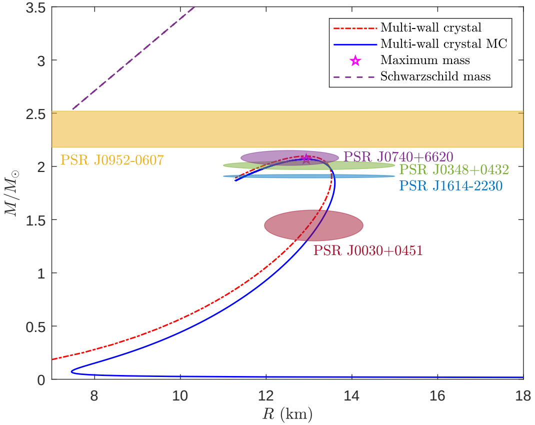

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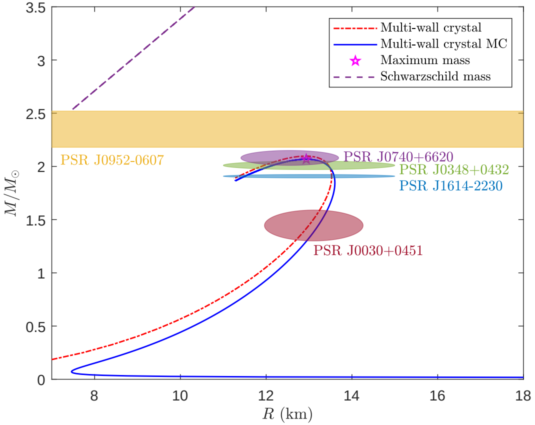

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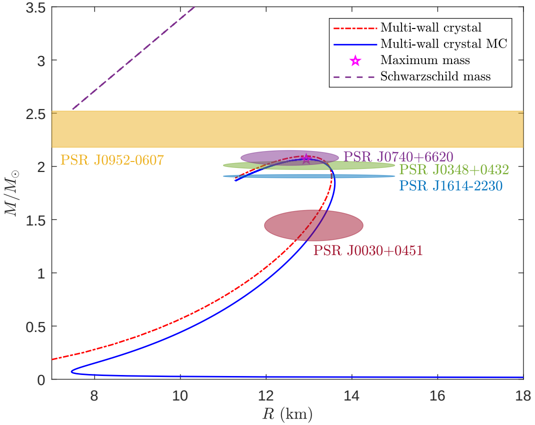

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$\Longrightarrow$ Resulting neutron stars agree well with recent NICER/LIGO observational data

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- Analogous to “pseudo-gap” phenomenon in condensed matter physics

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