Isospin asymmetric nuclear matter in the Skyrme model

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UNIVERSITY OF LEEDS

Outline of talk

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- Skyrme crystals and phases of skyrmion matter [Harland, Leask & Speight (2023) - arXiv:2305.14005]
- Applications of skyrmion crystals to dense nuclear matter [Leask, Huidobro & Wereszczynski (2023) - arXiv:2306.04533]

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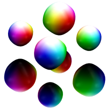

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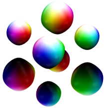
- Can we obtain a **single EoS** that yields neutron stars with crusts?
- Can these neutron stars have sufficient maximal masses?

Skyrme crystals and phases of skyrmion matter

Generalized Skyrme model



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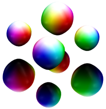

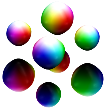

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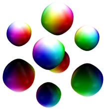
$$\Lsh L_\mu = \varphi^\dagger \partial_\mu \varphi \in \mathfrak{su}(2)$$



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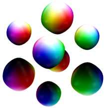

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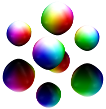

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- Baryon d.o.f. not explicitly visible $\to$ topology: Homotopy invariant $\leftrightarrow$ Baryon number

$$H_3(M) = \mathbb{Z} \ni B = \int_M \mathrm{d}^3 x \sqrt{-g}\, \mathcal{B}^0, \quad \mathcal{B}^\mu = \frac{1}{24\pi^2 \sqrt{-g}} \epsilon^{\mu\nu\rho\sigma} \mathrm{Tr}(L_\nu L_\rho L_\sigma)$$

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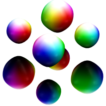
- Baryons realized as non-perturbative excitations of the pions $\Rightarrow$ solutions of the Euler–Lagrange field equations - <u>topological solitons</u> (**skyrmions**)



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$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\partial(\sqrt{-g}\mathcal{L}_{0246})}{\partial g^{\mu\nu}} \qquad \frac{\pi^4\lambda^2 e^4 F_\pi^2}{2\hbar^3} = c_6$$

$$= -\operatorname{Tr}(L_\mu L_\nu) - \frac{1}{4}g^{\alpha\beta}\operatorname{Tr}([L_\mu, L_\alpha][L_\nu, L_\beta]) + 2c_6\mathcal{B}_\mu\mathcal{B}_\nu + g_{\mu\nu}\mathcal{L}_{0246}$$

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- The adimensional static energy is thus ($T_{00} = \mathcal{E}_{\text{stat}} + \mathcal{E}_{\text{kin}}$)

$$\begin{aligned} M_B(\varphi, g) &= \int_M \mathrm{d}^3x \sqrt{-g}\, \mathcal{E}_{\text{stat}} \\ &= \int_M \mathrm{d}^3x \sqrt{-g} \left\{ -\frac{1}{2} g^{ij} \operatorname{Tr}(L_i L_j) - \frac{1}{16} g^{ik} g^{jl} \operatorname{Tr}\left([L_i, L_j][L_k, L_l]\right) \right. \\ m = \frac{2m_\pi}{F_\pi e} \to \quad & \left. + m^2 \operatorname{Tr}\left(\mathbb{I}_2 - \varphi\right) + c_6 \frac{\epsilon^{ijk}\epsilon^{abc}}{(24\pi^2\sqrt{-g})^2} \operatorname{Tr}(L_i L_j L_k) \operatorname{Tr}(L_a L_b L_c) \right\} \end{aligned}$$

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- We use the values

$$\boxed{F_\pi = 122\,\mathrm{MeV}, \quad e = 4.54, \quad m_\pi = 140\,\mathrm{MeV}, \quad \lambda^2 = 1\,\mathrm{MeV\,fm}^3}$$

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- Ground state of dense nuclear matter has a **crystalline** structure in the classical approximation
- In order to determine skyrmion crystals, we first need some numerical machinery!
- We will employ the usual vector (or σ-model) formulation and introduce the **metric independent integral formulation** (MIIF)

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$$+\sqrt{g}\left\{m^2\int_{\mathbb{T}^3} \mathrm{d}^3x\,\mathrm{Tr}\left(\mathbb{I}_2-\varphi\right)\right\} + \frac{1}{\sqrt{g}}\left\{c_6\frac{\epsilon^{ijk}\epsilon^{abc}}{(48\pi^2)^2}\int_{\mathbb{T}^3} \mathrm{d}^3x\,\mathrm{Tr}(L_i\Omega_{jk})\,\mathrm{Tr}(L_a\Omega_{bc})\right\}$$

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$\Rightarrow$ Define the metric independent integrals $L_{ij}(\varphi)$, $\Omega_{ijkl}(\varphi)$, $W(\varphi)$, $C(\varphi)$

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- In the vector formulation, the MII's are

$$W(\varphi) = 2m^2 \int_{\mathbb{T}^3} \mathrm{d}^3x\, (1 - \varphi^0)$$

$$L_{ij}(\varphi) = \int_{\mathbb{T}^3} \mathrm{d}^3x\, (\partial_i \varphi^\mu \partial_j \varphi^\mu)$$

$$\Omega_{ijkl}(\varphi) = \frac{1}{2} \int_{\mathbb{T}^3} \mathrm{d}^3x\, \{(\partial_i \varphi^\mu \partial_k \varphi^\mu)(\partial_j \varphi^\nu \partial_l \varphi^\nu) - (\partial_i \varphi^\mu \partial_l \varphi^\mu)(\partial_j \varphi^\nu \partial_k \varphi^\nu)\}$$

$$C(\varphi) = \frac{c_6}{(12\pi^2)^2} \int_{\mathbb{T}^3} \mathrm{d}^3x\, (\epsilon^{ijk} \epsilon_{\mu\nu\rho\sigma} \varphi^\mu \partial_i \varphi^\nu \partial_j \varphi^\rho \partial_k \varphi^\sigma)(\epsilon^{lmn} \epsilon_{\alpha\beta\gamma\delta} \varphi^\alpha \partial_l \varphi^\beta \partial_m \varphi^\gamma \partial_n \varphi^\delta)$$

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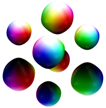

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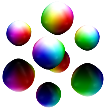

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- Fix Skyrme field to be the map $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$

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- The metric on $\mathbb{T}^3$ is the pullback $g = F^* g_{\mathrm{Euc}}$ with $g_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j$
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Skyrmion crystals

- Skyrme crystals are energy minimizing maps

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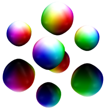

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- For fixed $\mathcal{L}_{024}$-field φ, there always **exists** a critical point of $M_B(\varphi, g)$ w.r.t. variations of g and it is in fact a **unique** c.p. (generalizes to $\mathcal{L}_{0246}$-model)

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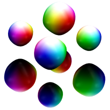

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- The $\varphi_{1/2}$-crystal [Kugler & Shtrikmann (1988)] can be obtained from a Fourier series-like expansion as an initial configuration [Castillejo *et al.* (1989)],

$$\varphi^0 = -c_1 c_2 c_3, \quad \varphi^1 = s_1 \sqrt{1 - \frac{s_2^2}{2} - \frac{s_3^2}{2} + \frac{s_2^2 s_3^2}{3}}, \quad \text{and cyclic,}$$

where $s_i = \sin(2\pi x^i/L)$ and $c_i = \cos(2\pi x^i/L)$, with initial metric $g = L^3 \mathbb{1}_3$.

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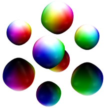

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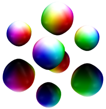

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$\Rightarrow$ **Multi-wall crystal** is an ideal candidate for **dense nuclear matter**

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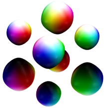

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- This is related to the static spatial part of $T_{\mu\nu}$: $S_{ij} = \frac{1}{\sqrt{g}}\frac{\delta(\sqrt{g}\mathcal{L}_{0246})}{\delta g^{ij}} = -\frac{1}{2}\,T_{ij}$

Extended virial constraints

- Space of allowed variations $\mathscr{E} = \{\delta g_{ij} \mathrm{d}x^i \mathrm{d}x^j \in \Gamma(\odot^2 T^* \mathbb{T}^3) : \delta g_{ij} \, \mathrm{const.}, \delta g_{ji} = \delta g_{ij}\}$

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- Second condition $S \perp_{L^2} \mathscr{E}_0$ coincides with the **extended virial constraints** derived by [Manton (2009)]

Extended virial constraints

- The Derrick scaling argument is

$$\int_{\mathbb{T}^3} \mathrm{d}^3x\sqrt{g}\,\langle S(\varphi,g),g\rangle_g = \int_{\mathbb{T}^3} \mathrm{d}^3x\sqrt{g}\,\mathrm{Tr}_g(S) = \frac{1}{2}\left(E_2 - E_4 + 3E_0 - 3E_6\right) = 0$$

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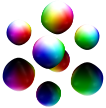

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- The Derrick scaling argument is

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- For a solution to be a skyrmion crystal it has to satisfy these extended virial constraints

Numerical minimization of the field and lattice

- Fix $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$ and think of the energy as a map $E_\varphi : \mathrm{SPD}_3 \to \mathbb{R}$ such that $E_\varphi := M_B(\varphi|_{\text{fixed}}, g)$

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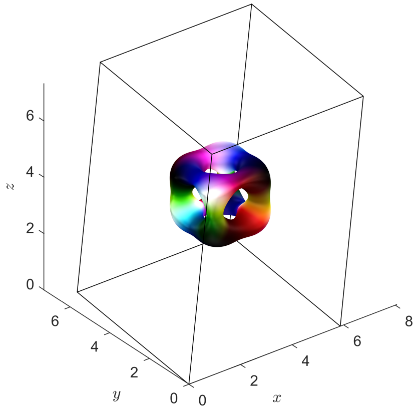
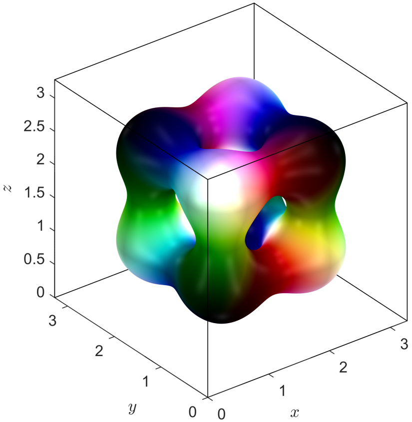
$\Rightarrow$ **Laddering of minimizations** as mentioned in Martin's talk

An example: the α-particle



(a) Initial configuration of a $B = 4$ RMA in a non-cubic lattice Λ



(b) Relaxed final solution of the cubic α-particle crystal

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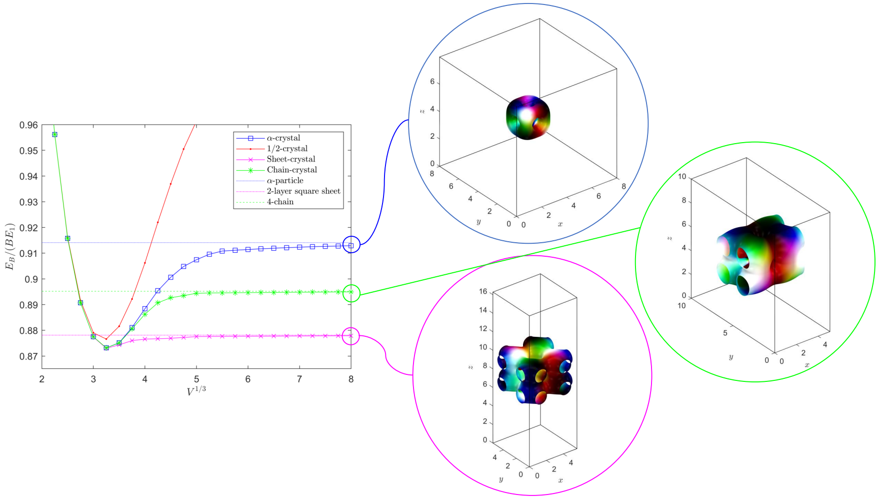

Quantum skyrmion crystals and the symmetry energy

Isospin quantization

- Skyrme model is non-renormalizable $\Rightarrow$ semi-classical quantization:
  $\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^{\dagger}(t)$ [Klebanov (1985)]

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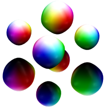

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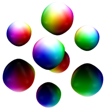

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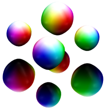

Isospin quantization

- Calculating the isospin correction to the energy of the crystal requires knowledge of the quantum state of the **whole crystal**
- The crystal is infinitely extended with infinite baryon number $\Rightarrow$ extremely difficult computation!
- The following restrictions are imposed [Adam *et al.* (2022)]:
  - $|\Psi\rangle = \otimes_{N_{\text{cell}}} |\psi\rangle$, where $N_{\text{cell}} \to \infty$ in the thermodynamic limit.
  - The symmetry of the classical configuration in each unit cell is extended to the whole crystal, so **both** wavefunctions share the same point symmetry group
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$$E_{I,I_3} = \frac{\hbar^2 I(I+1)}{N_{\text{cell}} U_{11}} + \frac{\hbar^2 I_3^2}{2}\left(\frac{1}{U_{33}} - \frac{2}{U_{11}}\right) \xrightarrow[N_{\text{cell}}\to\infty]{} E_{\text{iso}} = \frac{E_{I,I_3}}{N_{\text{cell}}} = \frac{\hbar^2}{8\,U_{33}} B_{\text{cell}}^2 \delta^2$$

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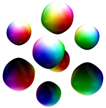

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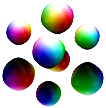

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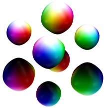

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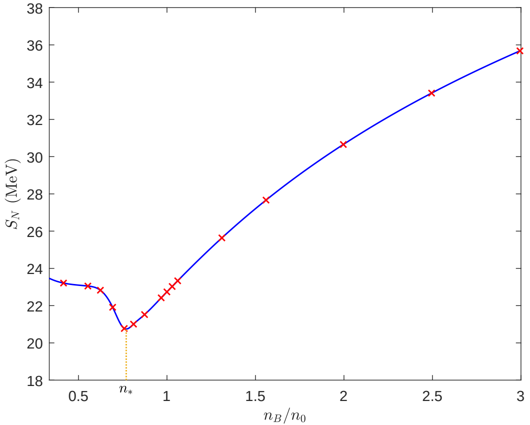
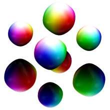
- At saturation we find $n_0 = 0.160\,\mathrm{fm}^{-3}$, $E_N(n_0) = 912\,\mathrm{MeV}$ and $S_N(n_0) = 22.7\,\mathrm{MeV}$

Symmetry energy and the cusp structure





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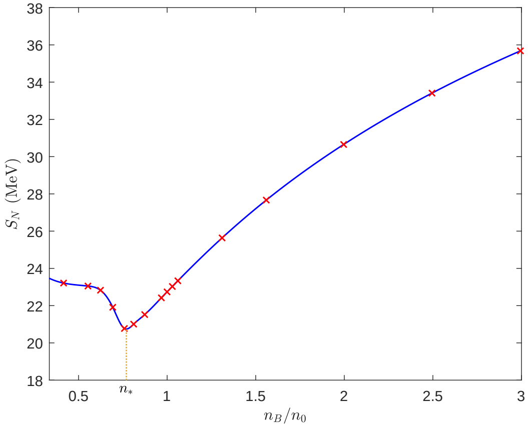

- **Cusp** below saturation at $n_* \sim 3n_0/4$

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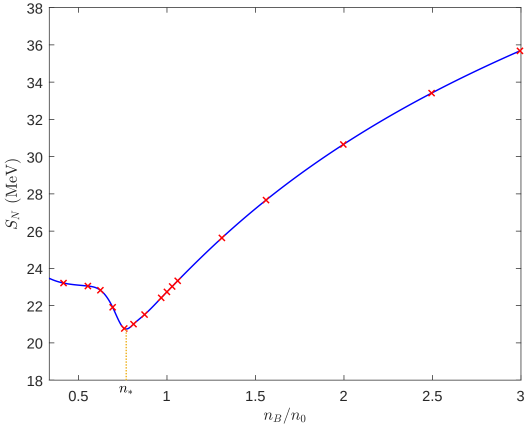

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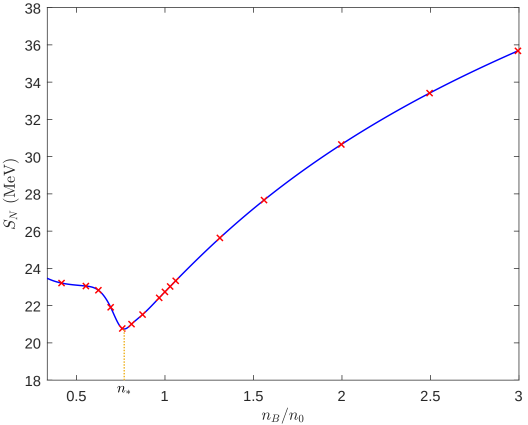

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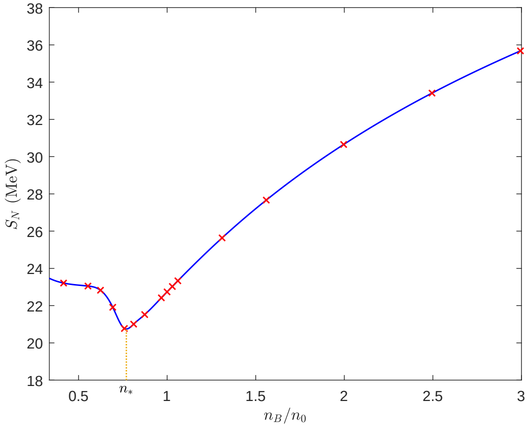

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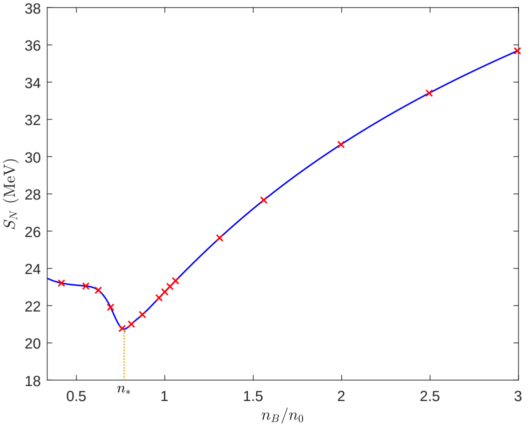

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- Cusp origin: **phase transition** between **infinite isospin asymmetric nuclear matter** and somewhat **isolated finite nuclear matter** [P.L., M.H. & A.W. (2023)]

Particle fractions of $npe\mu$ matter in β-equilibrium

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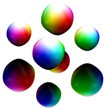

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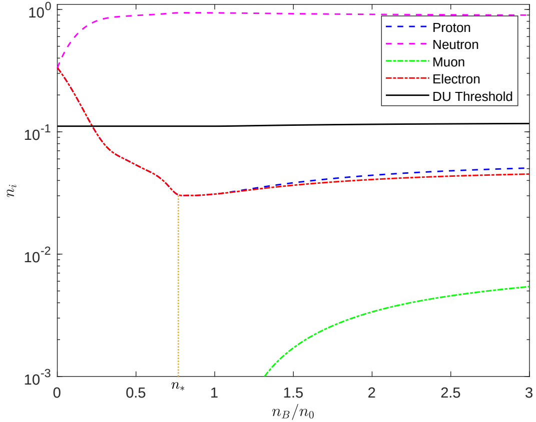

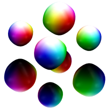

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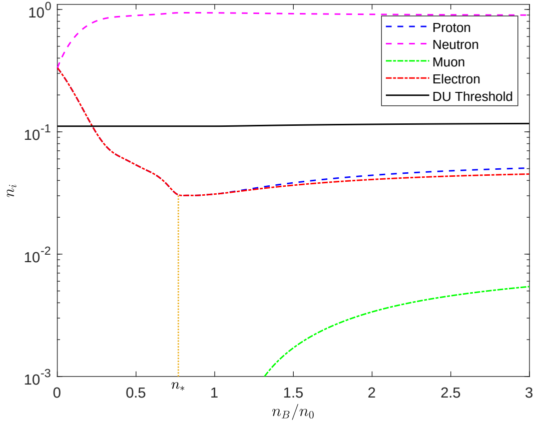

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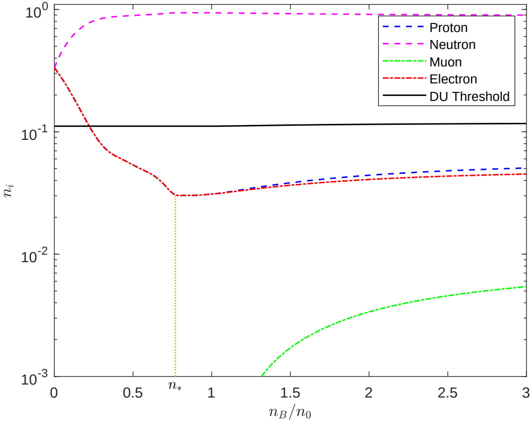

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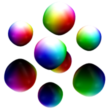

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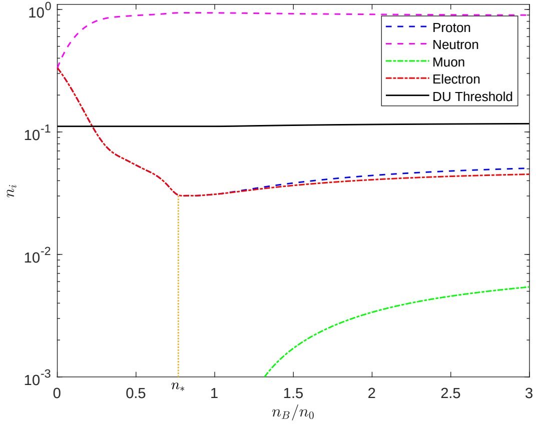

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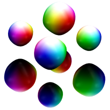

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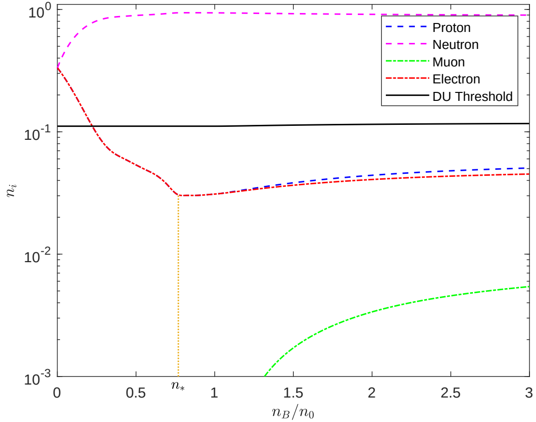

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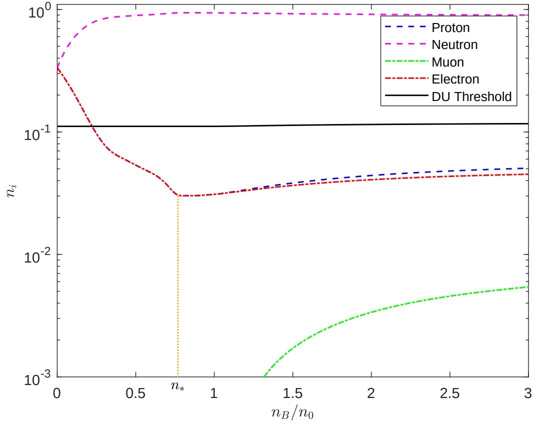

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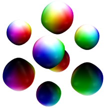

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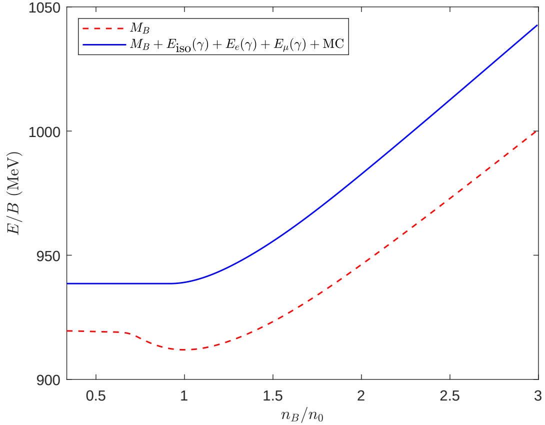
Isospin asymmetric equation of state



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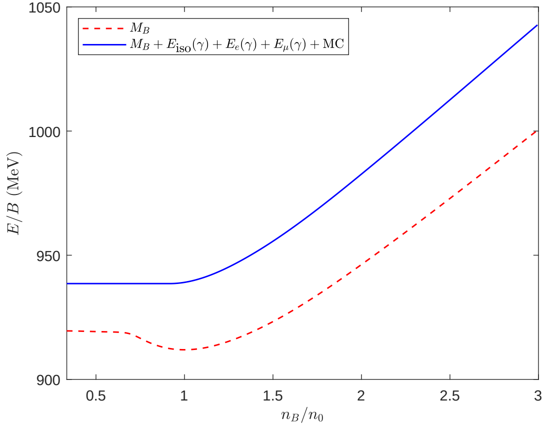

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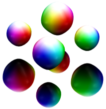

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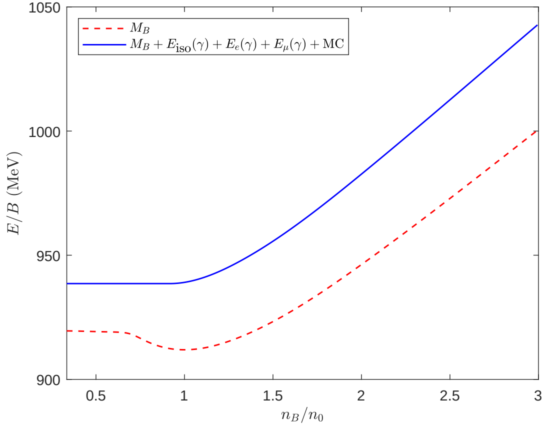

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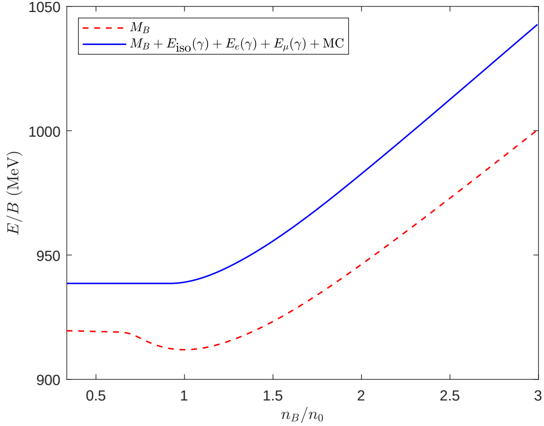

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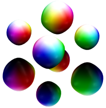
- We will use this EoS to obtain NS within the Skyrme model



Neutron stars

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- The energy density ρ and the pressure p are related by the (multi-wall) crystal EoS $\rho(p) = \rho_{\mathrm{MW}}(p)$ [Adam *et al.* (2020)]

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- Substituting this into the Einstein equations $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$ yields the TOV system

$$\frac{\mathrm{d}A}{\mathrm{d}r} = A(r)r\left(8\pi G B(r)p(r) - \frac{1-B(r)}{r^2}\right)$$

$$\frac{\mathrm{d}B}{\mathrm{d}r} = B(r)r\left(8\pi G B(r)\rho(p(r)) + \frac{1-B(r)}{r^2}\right)$$

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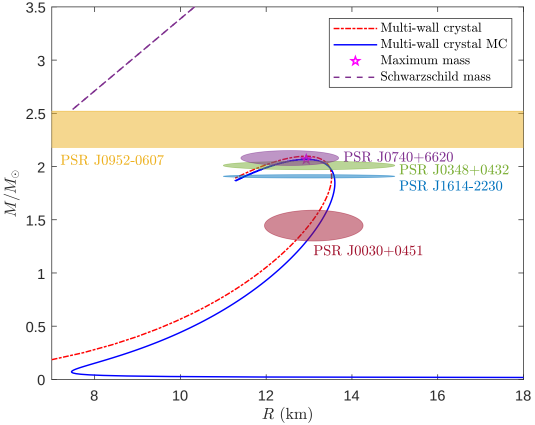
Neutron star properties and the mass-radius curve



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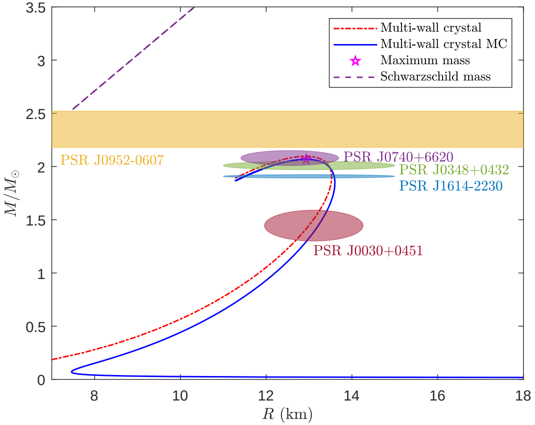

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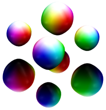

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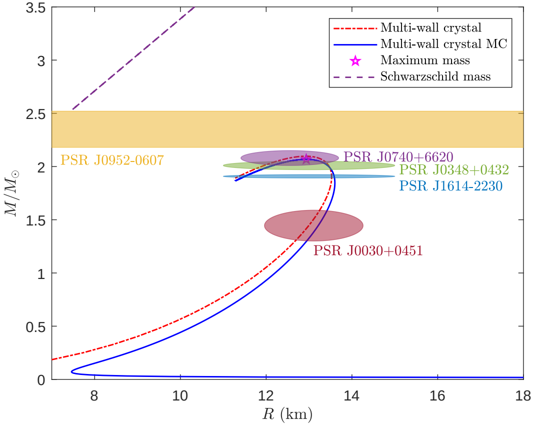

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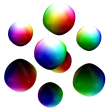

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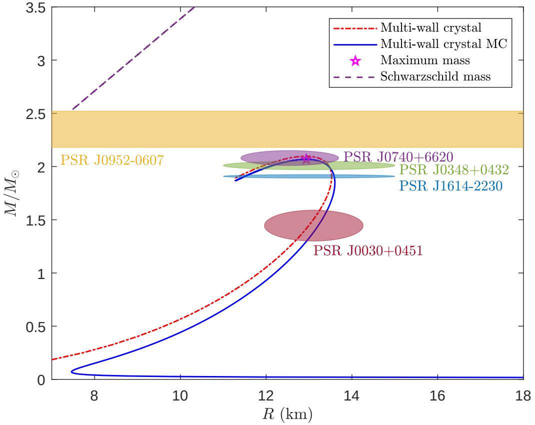

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Towards the semi-empirical mass formula (SEMF)

α-particle approximation (APA)

- Bethe–Weizsäcker SEMF:

$$E_b = a_V B - a_S B^{2/3} - a_C \frac{Z(Z-1)}{B^{1/3}} - a_A \delta^2 B + \delta(N, Z)$$

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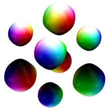

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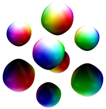

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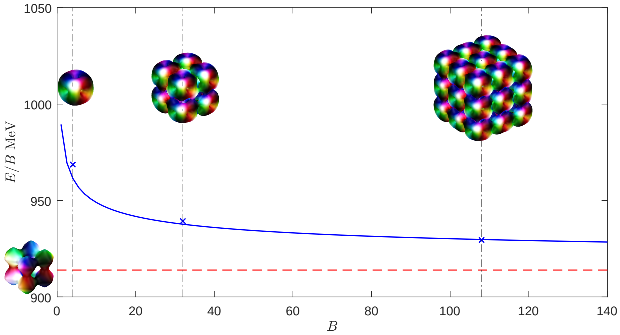

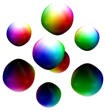

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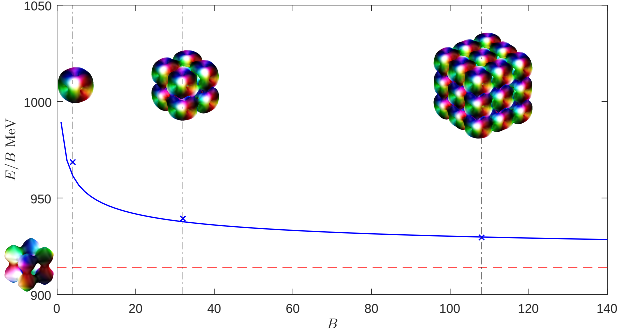

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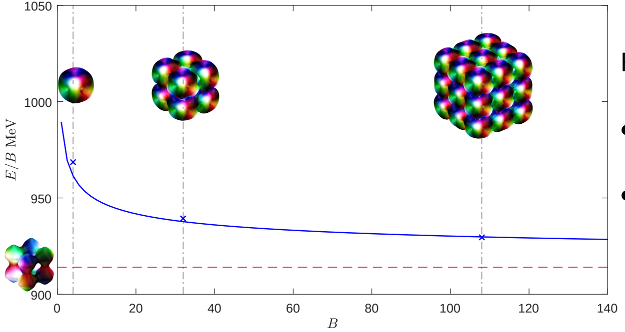

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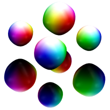

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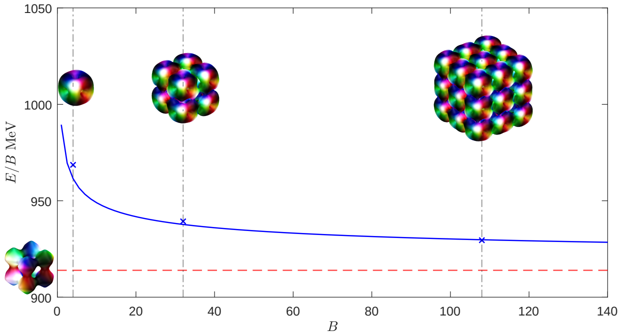

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