New Skyrme Crystals

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Motivation

Motivation

- The Skyrme model is a nonlinear field theory of pions (Skyrme, 1961)
- Nuclei are modelled as topological solitons (Skyrmions)
- Want to understand phases, and transitions of phases, of nuclear matter in the Skyrme model
- Ground state of nuclear matter has a crystalline structure in the classical approximation
- Many Skyrmions look like chunks of the infinite crystal (Feist *et al.*, 2013)
- Two candidates proposed:
  - Cubic lattice of half-Skyrmions
  - α-particle lattice
- Which is the lower energy solution (classically)?
- Are the two related?
- Are there other *new* solutions?

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Skyrme model

Skyrme model

- Topological solitons: smooth, spatially localized solutions of non-linear field theories, topologically stable against decay to vacuum.
- Skyrme field $\varphi : (\Sigma, g) \to (G, h)$, e.g. $\mathbb{R}^3/\Lambda \to \mathrm{SU}(2)$
  - Left-invariant Maurer-Cartan form $\theta = \varphi^{-1}\mathrm{d}\varphi \in \Omega^1(G) \otimes \mathfrak{g}$
  - Associated 2-form $\Omega \in \Omega^2(G) \otimes \mathfrak{g}$, $\Omega(X, Y) := [\theta(X), \theta(Y)]$
  - $\mathrm{Ad}(\mathrm{SU}(2))$ invariant inner product, $h(X, Y) = \frac{1}{2}\,\mathrm{Tr}(X^\dagger Y)$
  - Skyrme energy functional

$$E[\varphi] = \int_\Sigma \left\{ c_2 g^{ij} h(L_i, L_j) + \frac{c_4}{2} g^{ia} g^{jb} h(\Omega_{ij}, \Omega_{ab}) + c_0 V(\varphi) \right\} \mathrm{vol}_g$$

  - $V : \mathrm{SU}(2) \to \mathbb{R}$ is the pion mass potential,

$$V(\varphi) = m^2\,\mathrm{Tr}(\mathbb{1} - \varphi)$$

  - Usual coupling constants $c_0 = c_2 = 1$ and $c_4 = 1/4$
  - Derrick's scaling argument $E_4 = E_2 + 3E_0$
  - Topological energy bound $E \geq 12\pi^2\alpha|B|$ (Harland, 2014)
- Skyrmions are local minima of E
- Found numerically by discretising E and applying a gradient descent method

History of Skyrme crystals

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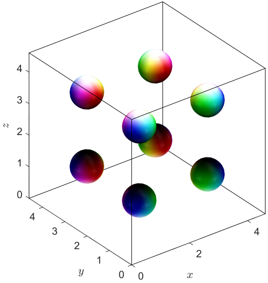
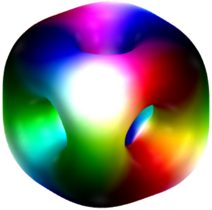
History of Skyrme crystals



(a) Half-Skyrmions lattice



(b) α-particle

- SC crystal of Skyrmions (Klebanov, 1985)
- BCC crystal of half Skyrmions (Goldhaber & Manton, 1987)
- SC crystal of half Skyrmions (Kugler & Shtrikman, 1988; Castillejo *et al.*, 1989)
- Building Skyrmions from the α-particle (Battye *et al.*, 2007)
- Massless (Silva Lobo, 2010) and massive (Adam *et al.*, 2022) phase transition between α-particle and SC crystal of half Skyrmions
- Constructing Skyrmions from crystal chunks (Feist *et al.*, 2013)
- Phase transitions between different crystals (Perapechka & Shnir, 2017)

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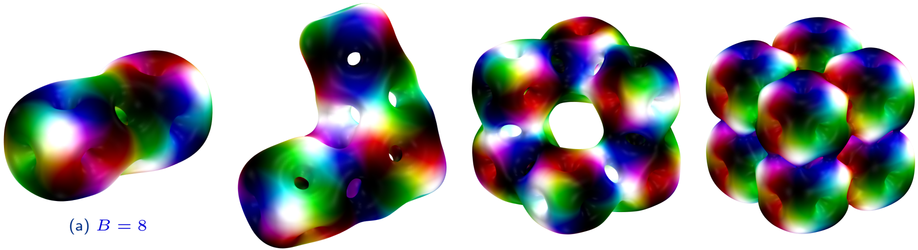
History of Skyrme crystals



(a) $B = 8$ (b) $B = 12$ (c) $B = 24$ (d) $B = 32$

Figure: Skyrmions constructed from the α-particle

(a) Constructed from two twisted α-particles (Battye *et al.*, 2007)
(b) Constructed from three twisted α-particles (Battye *et al.*, 2007)
(c) Constructed from six twisted α-particles (Feist *et al.*, 2013)
(b) Constructed from eight α-particles (Battye *et al.*, 2007)

Variational problem

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Variational problem

- Warmup problem: Baby Skyrmion crystals (Leask, 2022)
- Skyrme crystals are maps

$$\varphi : \mathbb{R}^3/\Lambda \to \mathrm{SU}(2), \quad \Lambda = \{n_1\mathbf{X}_1 + n_2\mathbf{X}_2 + n_3\mathbf{X}_3 : n_i \in \mathbb{Z}\}$$

- General idea (Speight, 2014): identify $(\mathbb{R}^3/\Lambda, \bar{g}) \longleftrightarrow (\mathbb{R}^3/\mathbb{Z}^3, g)$ via the diffeomorphism $F : \mathbb{T}^3 \to \mathbb{R}^3/\Lambda$ where $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$ and $F(\mathbf{x}) = \mathcal{A}\mathbf{x}, \mathcal{A} = [\mathbf{X}_1\, \mathbf{X}_2\, \mathbf{X}_3]$
- Fix Skyrme field to be the map $\varphi : \mathbb{T}^3 \to \mathrm{SU}(2)$
- Metric on $\mathbb{T}^3$ is the pullback $g = F^*\bar{g}$, with $g_{ij} = \mathbf{X}_i \cdot \mathbf{X}_j$
- Vary metric g_s with $g_0 = F^*\bar{g} \Longleftrightarrow$ vary lattice Λ_s with $\Lambda_0 = \Lambda$
- Energy minimized over all variations of $g \Longleftrightarrow$ optimal period lattice Λ

Variational problem

- Let SPD_n be the space of symmetric positive-definite $n \times n$-matrices.
- For fixed φ, can consider the Skyrme energy to be a map $E : \mathrm{SPD}_3 \to \mathbb{R}$
- E is convex when restricted to geodesics $\Rightarrow$ uniqueness of the lattice
- Simple case: $\Lambda = L\mathbb{Z}^3$, energy scales as $E = LE_2 + \frac{1}{L}E_4 + L^3E_0$ and

$$\frac{\mathrm{d}E}{\mathrm{d}L} = 0 \quad \Rightarrow \quad L^2 = \frac{1}{2}\left(-\frac{E_2}{3E_0} + \sqrt{\left(\frac{E_2}{3E_0}\right)^2 + \frac{4E_4}{3E_0}}\right)$$

- In general, optimal period lattice Λ:
  - ◇ Massless pions: explicit solution
    - ↬ Matrix square root
  - ◇ Massive pions: numerical solution
    - ↬ Arrested Newton flow or nonlinear conjugate gradient descent

Matrix square root

- Massless case $V(\varphi)=0$ reduces the problem to

$$\left(\frac{g\mathcal{K}}{\sqrt{\det g}}\right)^2=\frac{c_2}{c_4}\mathcal{L}\mathcal{K},$$

  where, in sigma model notation,

$$\mathcal{L}_{ij}=\int_\Sigma(\partial_i\pi\cdot\partial_j\pi)\,\mathrm{vol}_g$$

  and

$$\mathcal{K}^{ij}=\varepsilon^{iab}\varepsilon^{jcd}\int_\Sigma\{(\partial_a\pi\cdot\partial_c\pi)(\partial_b\pi\cdot\partial_d\pi)-(\partial_a\pi\cdot\partial_d\pi)(\partial_b\pi\cdot\partial_c\pi)\}\,\mathrm{vol}_g$$

- Matrix square root:

$$\frac{g\mathcal{K}}{\sqrt{\det g}}=PD^{1/2}P^{-1},\quad D^{1/2}=\begin{pmatrix}\sqrt{\lambda_1}&0&0\\0&\sqrt{\lambda_2}&0\\0&0&\sqrt{\lambda_3}\end{pmatrix}$$

- Setting $\tilde{g}=\frac{g}{\sqrt{\det g}}$ yields $\tilde{g}=PD^{1/2}P^{-1}\mathcal{K}^{-1}$
- Finally, using the fact that $\det g=\frac{1}{(\det\tilde{g})^2}$, we arrive at $g=\frac{\tilde{g}}{\det\tilde{g}}$.

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Numerical approach to the lattice

- Aim: solve the unconstrained optimisation problem

$$\min_{g \in \mathrm{SPD}_3} E(g)$$

- Accelerated 2nd order gradient descent with flow arresting
- Solve Newton's equations of motion for a particle on SPD_3 with potential energy $E(g)$ using 4th order Runge–Kutta:

$$\partial_{ss} g_s\big|_{s=0} = -\left.\frac{\delta E}{\delta g_s}\right|_{s=0}, \quad g_0 = g$$

- Restart flow if $E(t+\delta t) > E(t)$ (arresting)
- Terminate flow when $\left.\frac{\delta E}{\delta g_s}\right|_{s=0} < 10^{-5}$ everywhere

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Massless Skyrme crystals

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$1/2$-lattice (FCC lattice)

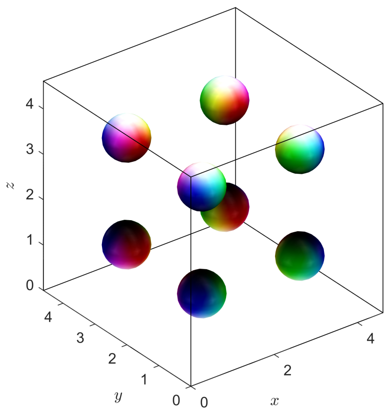

Figure: Massless FFC lattice (Kugler & Shtrikman, 1988) and (Castillejo *et al.*, 1989)

- Obtained from Fourier series expansion initial configuration (Castillejo *et al.*, 1989)

$$\sigma = -c_1 c_2 c_3, \quad \pi^1 = s_1 \sqrt{1 - \frac{s_2^2}{2} - \frac{s_3^2}{2} + \frac{s_2^2 s_3^2}{3}}$$

  and cyclic, where $c_i = \cos\left(\frac{2\pi x^i}{L}\right)$ and $s_i = \sin\left(\frac{2\pi x^i}{L}\right)$

- Symmetries (Kugler & Shtrikman, 1989):

$$\begin{aligned}
A_1 &: (x,y,z) \mapsto (-x,y,z) \\
&\quad (\sigma,\pi^1,\pi^2,\pi^3) \mapsto (\sigma,-\pi^1,\pi^2,\pi^3) \\
A_2 &: (x,y,z) \mapsto (y,z,x) \\
&\quad (\sigma,\pi^1,\pi^2,\pi^3) \mapsto (\sigma,\pi^2,\pi^3,\pi^1) \\
C_3 &: (x,y,z) \mapsto (x,z,-y) \\
&\quad (\sigma,\pi^1,\pi^2,\pi^3) \mapsto (\sigma,\pi^1,\pi^3,-\pi^2) \\
D_4 &: (x,y,z) \mapsto (x+L/2,y,z) \\
&\quad (\sigma,\pi^1,\pi^2,\pi^3) \mapsto (-\sigma,-\pi^1,\pi^2,\pi^3)
\end{aligned}$$

New crystals from old

- Define a potential $V(Q) := E[Q\pi_{1/2}]$ for $Q \in \mathrm{SO}(4)$
- Isospin symmetry group of the $1/2$-lattice is

$$\Gamma = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

- Then $V(QG) = V(Q)\,\forall\, G \in \Gamma$ and $V(SQ) = V(Q)\,\forall\, S \in \mathrm{SO}(3)$
- So the potential is a map $V : \mathrm{SO}(4)/\,\mathrm{SO}(3) \cong S^3 \to \mathbb{R}$
- N/S-pole on S^3 corresponds to the $1/2$-lattice
- Cubic symmetry group acts invariantly on equatorial S^2
- Intersection of S^2 with vertices, edge centres & face centres of the cube projected radially onto S^2 are critical points of V (principle of symmetric criticality)
- *Three* other Skyrme crystals should exist, one of which is the α-lattice

$\Rightarrow$ Two entirely new Skyrme crystals

New crystals from old

- Let the map $\mathrm{SO}(4)/\mathrm{SO}(3) \cong S^3$ be given explicitly by

$$Q \mapsto Q^T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \mathbf{v}_1, \quad Q^T = \begin{pmatrix} \uparrow & \uparrow & \uparrow & \uparrow \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{v}_4 \\ \downarrow & \downarrow & \downarrow & \downarrow \end{pmatrix}$$

- Let $\{\mathbf{v}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4\}$ be a set of linearly independent vectors in $\mathbb{R}^4$.
- Use Gram-Schmidt to obtain an orthonormal basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ for $\mathbb{R}^4$
- Can explicitly construct a Q such that $\pi = Q\pi_{1/2}$

$\Rightarrow$ The three additional crystals can easily be obtained

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4-lattice (α-lattice)

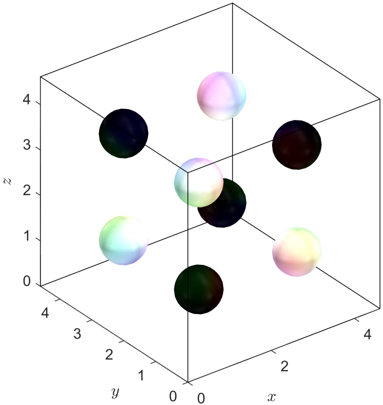


Figure: Massless α-particle lattice

- Obtained from the $B = 4$ rational map initial configuration (Houghton *et al.*, 1997)

$$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$$

- Already shown the α-lattice and FCC lattice are related by an $\mathrm{SO}(4)$-isospin transformation (Leask (SIG X Talk), 2022):

$$Q\pi_4 = \pi_{1/2}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1/\sqrt{3} & -1/\sqrt{6} & 1/\sqrt{2} & 0 \\ 1/\sqrt{3} & -1/\sqrt{6} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 2/\sqrt{6} & 0 & 0 \end{pmatrix}$$

4-lattice (α-lattice)

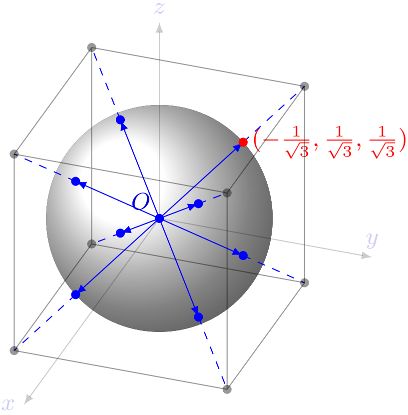
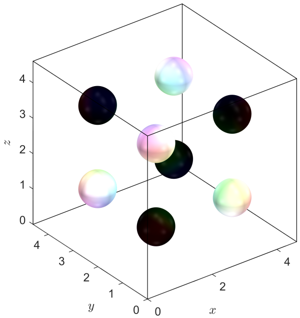


Figure: Cube vertices projected radially onto S^2

- Related to the FCC lattice by $\mathrm{SO}(4)$-isospin transformation:

$$\pi_4 = Q\pi_{1/2}, \quad Q = \begin{pmatrix} 0 & -1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \\ 0 & -1/\sqrt{2} & -1/\sqrt{2} & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$



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(New) 1-lattice

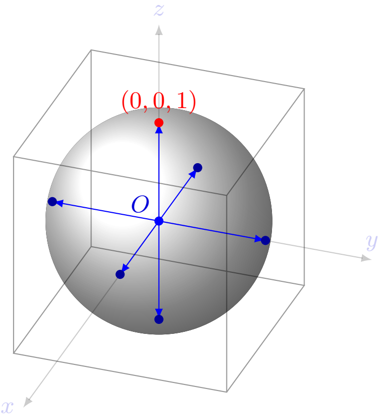
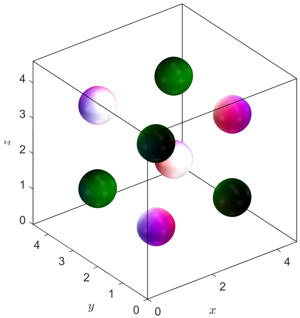
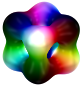


Figure: Cube face centres projected radially onto S^2

- Related to the FCC lattice by $\mathrm{SO}(4)$-isospin transformation:

$$\pi_1 = Q\pi_{1/2}, \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} & 0 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \end{pmatrix}$$





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(New) 2-lattice

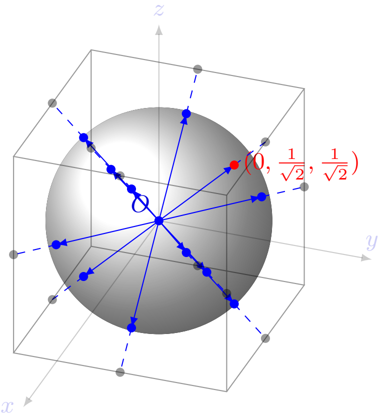
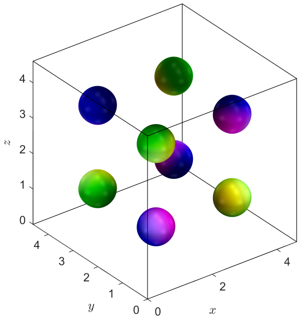


Figure: Cube edge centres projected radially onto S^2

- Related to the FCC lattice by $\mathrm{SO}(4)$-isospin transformation:

$$\pi_2 = Q\pi_{1/2}, \quad Q = \begin{pmatrix} 0 & 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 & 0 \\ 0 & 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{pmatrix}$$



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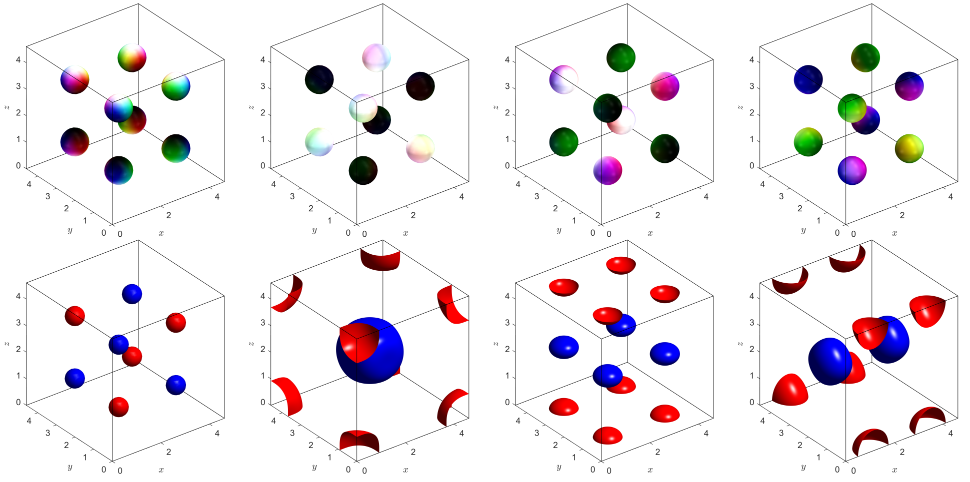

(a) $1/2$-lattice (b) 4-lattice (c) 1-lattice (d) 2-lattice

Figure: Skyrme crystals for $m = 0$. Top row is baryon density. Bottom row is sigma plots, where $\sigma = 0.9$ and $\sigma = -0.9$.

Massive Skyrme crystals

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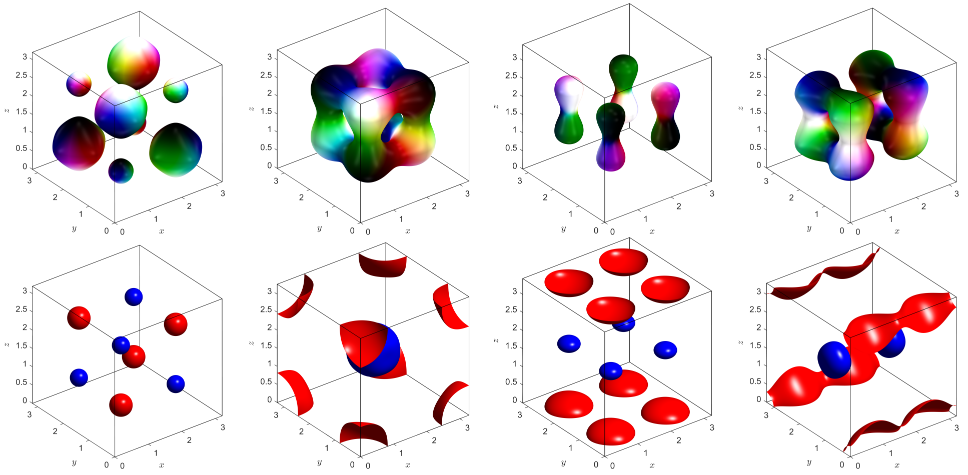

(a) $1/2$-lattice (b) 4-lattice (c) 1-lattice (d) 2-lattice

Figure: Skyrme crystals for $m = 1$. Top row is baryon density. Bottom row is sigma plots, where $\sigma = 0.9$ and $\sigma = -0.9$.

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Isospin moment of inertia tensor

- Collective coordinate approach to isospin d.o.f. (Adkins *et al.*, 1983)

$$\varphi(x) \mapsto \hat{\varphi}(x,t) = A(t)\varphi(x)A^\dagger(t).$$

  Isorotations are symmetries of E so these configurations are all energy-degenerate.
- Isorotational angular velocity is $\omega_j = -i\,\mathrm{Tr}(\tau^j A^\dagger \dot{A})$
- Maurer-Cartan form transforms as

$$\hat{L}_\mu = \hat{\varphi}^\dagger \partial_\mu \hat{\varphi} = \begin{cases} A\omega_i T_i A^\dagger, & \mu = 0 \\ AL_i A^\dagger, & \mu = i = 1,2,3. \end{cases}$$

- $T_i = \frac{i}{2}\varphi^\dagger[\tau^i, \varphi]$ is also an $\mathfrak{su}(2)$ current
- Effective Lagrangian on restricted space of configurations is $L_{\text{eff}} = L_{\text{rot}} - M_B$, where

$$L_{\text{rot}} = \frac{1}{2}\omega_i U_{ij}\omega_j$$

  and the isospin moment of inertia is

$$U_{ij} = -\int_\Sigma \mathrm{Tr}\left(c_2 T_i T_j + c_4 g^{ab}[L_a, T_i][L_b, T_j]\right)\mathrm{vol}_g$$

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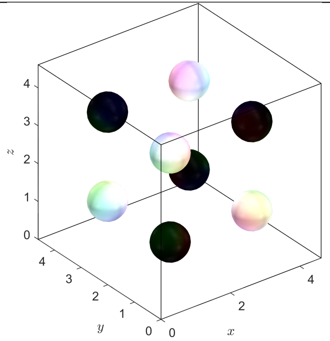
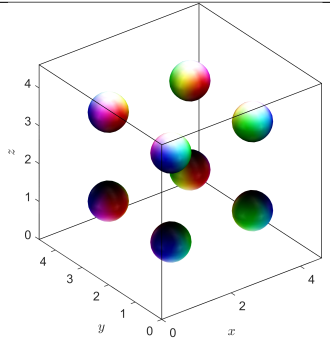
4-lattice	$1/2$-lattice
	
$E^4 = 1.0378$	$E^{1/2} = 1.0378$
$U^4 = \begin{pmatrix} 238.6 & 0 & 0 \\ 0 & 238.6 & 0 \\ 0 & 0 & 297.9 \end{pmatrix}$	$U^{1/2} = \begin{pmatrix} 297.9 & 0 & 0 \\ 0 & 297.9 & 0 \\ 0 & 0 & 297.9 \end{pmatrix}$

Table: Comparison of the massless ($m = 0$) 4-lattice and $1/2$-lattice

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New massless Skyrme crystals

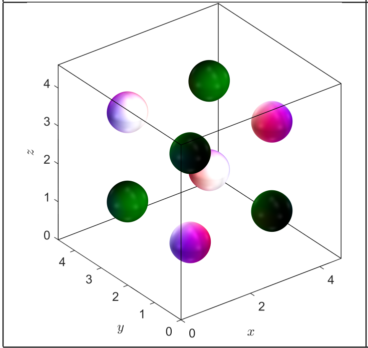
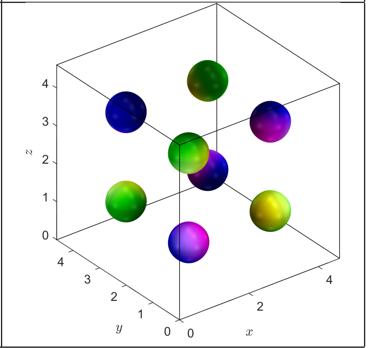
1-lattice	2-lattice
	
$E^1 = 1.0378$	$E^2 = 1.0378$
$U^1 = \begin{pmatrix} 238.6 & 0 & 0 \\ 0 & 238.6 & 0 \\ 0 & 0 & 297.9 \end{pmatrix}$	$U^2 = \begin{pmatrix} 238.6 & 0 & 0 \\ 0 & 238.6 & 0 \\ 0 & 0 & 297.9 \end{pmatrix}$

Table: Comparison of the massless $(m = 0)$ 1-lattice and 2-lattice

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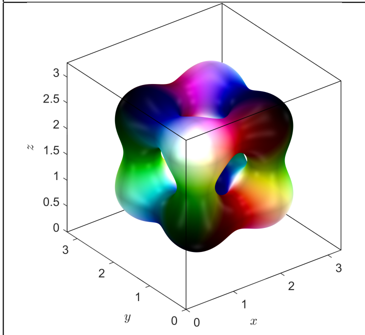
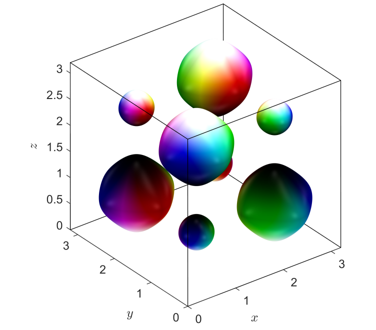
4-lattice	$1/2$-lattice
	
$E^4 = 1.0631$	$E^{1/2} = 1.0673$
$U^4 = \begin{pmatrix} 135.5 & 0 & 0 \\ 0 & 135.5 & 0 \\ 0 & 0 & 167.3 \end{pmatrix}$	$U^{1/2} = \begin{pmatrix} 165.2 & 0 & 0 \\ 0 & 165.2 & 0 \\ 0 & 0 & 165.2 \end{pmatrix}$

Table: Comparison of the massive ($m = 1$) 4-lattice and $1/2$-lattice

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New massive Skyrme crystals

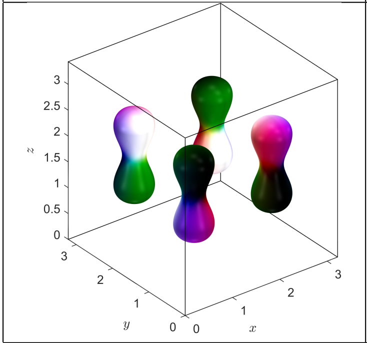
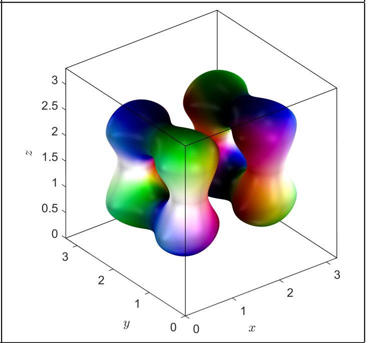
1-lattice	2-lattice
	
$E^1 = 1.0629$	$E^2 = 1.0631$
$U^1 = \begin{pmatrix} 166.8 & 0 & 0 \\ 0 & 135.8 & 0 \\ 0 & 0 & 135.8 \end{pmatrix}$	$U^2 = \begin{pmatrix} 167.2 & 0 & 0 \\ 0 & 135.7 & 0 \\ 0 & 0 & 135.6 \end{pmatrix}$

Table: Comparison of the massive ($m = 1$) 1-lattice and 2-lattice

Comparison of the four crystals

m	E^4	$E^{1/2}$	E^1	E^2
0	1.0378	1.0378	1.0378	1.0378
1	1.0631	1.0673	1.0629	1.0631
3	1.0710	1.0797	1.0703	1.0708
5	1.0715	1.0816	1.0707	1.0713
10	1.0710	1.0824	1.0703	1.0707

Table: Comparison of the four crystals for various m

- Massless crystals:
  - Energy degenerate
  - Isospin tensors share common eigenvalue(s)
  - All related to one another by $\mathrm{SO}(4)$-isospin transformations
- Massive crystals:
  - Energy is not invariant under $\mathrm{SO}(4)$ action
  - $\Rightarrow$ New lower energy crystal – the 1-lattice

Open problems

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Open problems

- What happens when we consider quantum corrections from the isospin d.o.f. for these two new crystals? (Adam *et al.*, 2022) investigated this for the $1/2$-lattice and the 4-lattice.
- How do the energies of the new 1-lattice and 2-lattice compare to the 4-lattice at high and low density (e.g. produce the $E(V)$ curve)?
  - New lattices are not cubic, do they scale with the pion mass?
- How can we construct Skyrmions from these new lattices?