Planar Skyrmion crystals

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Outline

Planar Skyrme model

The general planar Skyrme model consists of a single scalar field $\varphi : \Sigma \to S^2$ where (Σ, g) is a Riemannian manifold, and (S^2, h, ω) is the 2-sphere with round metric h and area 2-form ω.

We are interested in planar Skyrmions on:

- the plane, $\Sigma = \mathbb{R}^2$;
- the cylinder, $\Sigma = S^1 \times \mathbb{R}$; and
- the lattice, $\Sigma = \mathbb{R}^2/\Lambda$.

The static energy functional on Σ is given by

$$E[\varphi] = \int_\Sigma \left\{ \frac{1}{2}|\mathrm{d}\varphi|^2 + \frac{\kappa^2}{2}|\varphi^*\omega|^2 + V[\varphi] \right\} \mathrm{vol}_g.$$

The planar Skyrme map has an associated degree

$$B[\varphi] = -\frac{1}{4\pi}\int_\Sigma \varphi^*\omega \in \mathbb{Z},$$

and we refer to minimisers of the energy E for fixed charge B as Skyrmions.

Planar Skyrmions on $\mathbb{R}^2$

Consider the plane $\Sigma = \mathbb{R}^2$ with local coordinates $x = (x_1, x_2)$ and flat Euclidean metric $g = \mathrm{diag}(1, 1)$. The static energy functional takes the familiar form

$$E[\varphi] = \int_{\mathbb{R}^2} \left\{ \frac{1}{2} \partial_i \varphi \cdot \partial_i \varphi + \frac{\kappa^2}{4} \left(\partial_i \varphi \times \partial_j \varphi \right) \cdot \left(\partial_i \varphi \times \partial_j \varphi \right) + V[\varphi] \right\} \mathrm{d}x^1 \, \mathrm{d}x^2.$$

We consider two distinct potentials: the standard potential (Piette *et al.*, 1995)

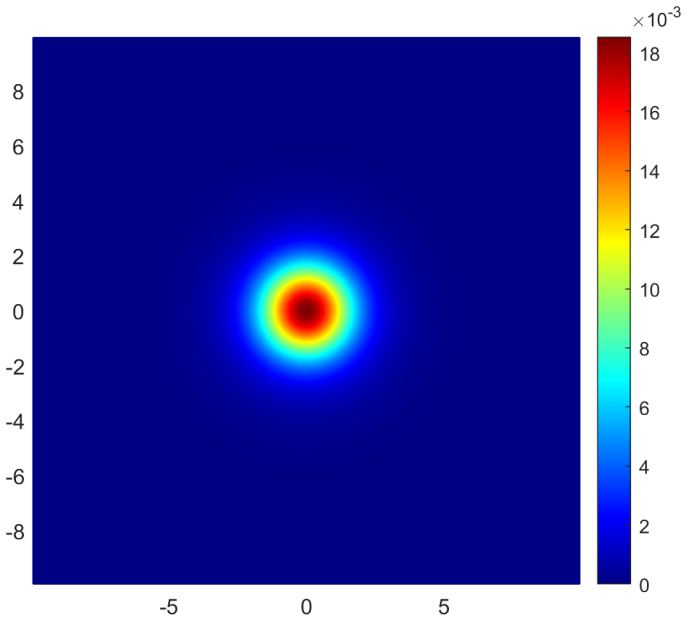
$$V[\varphi] = m^2 (1 - \varphi^3)$$

and the easy plane potential (Jäykkä & Speight, 2010)

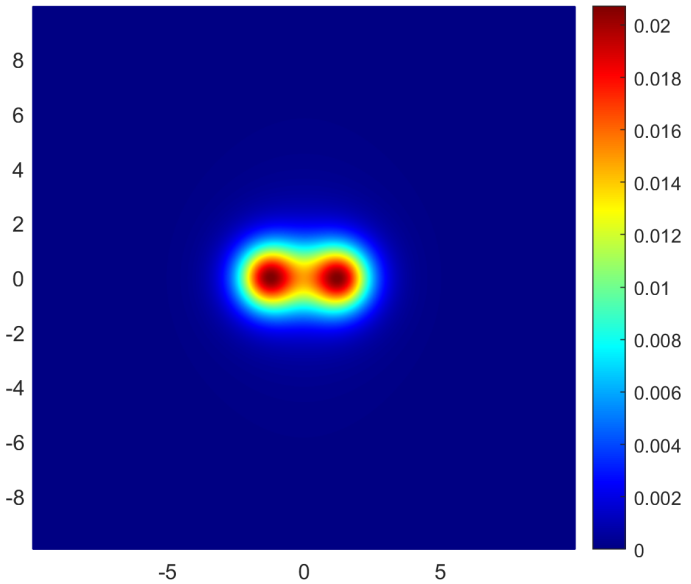
$$V[\varphi] = \frac{1}{2} m^2 (\varphi^1)^2.$$

Finite energy solutions require us to impose the boundary conditions $\lim_{|x| \to \infty} \varphi(x) \equiv \varphi_\infty = (0, 0, 1)$ such that $V[\varphi_\infty] = 0$.

$B = 1$ planar Skyrmions on $\mathbb{R}^2$



(a) Standard charge-1 energy density.



(b) Easy plane charge-1 energy density.

Figure 1: Plots of the energy density of (a) the axially symmetric charge-1 planar Skyrmion, for the standard potential $V[\varphi] = m^2(1 - \varphi^3)$, and (b) the charge-1 planar Skyrmion for the easy plane potential $V[\varphi] = \frac{1}{2}m^2(\varphi^1)^2$.

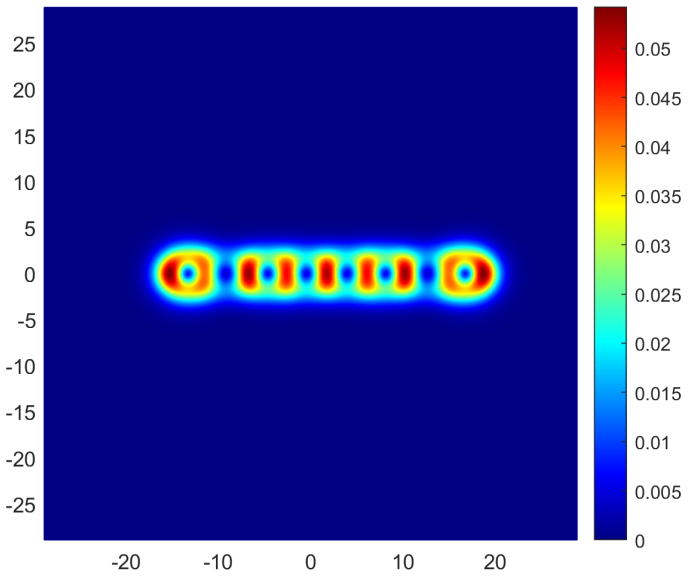
Standard baby Skyrmions on $\mathbb{R}^2$

Chains solutions were proposed as a good candidate for the global minima for low charges (Foster, 2010).

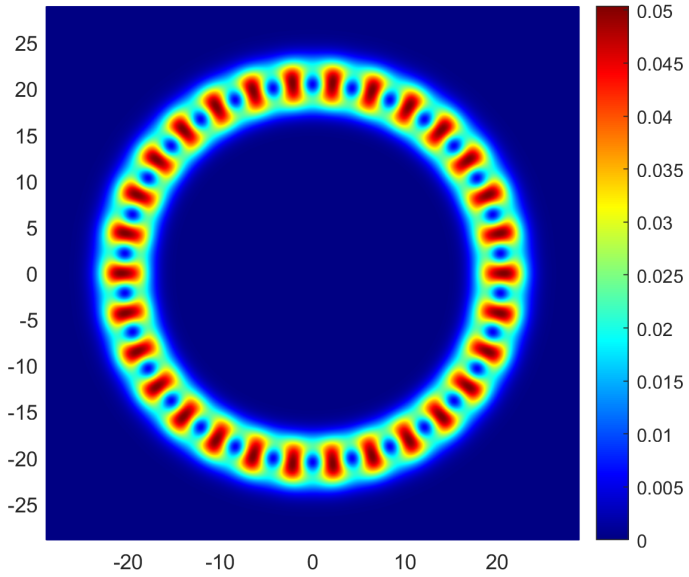
Ring solutions were found to be a better candidate for the global minima for charges $B > B_c \in \mathbb{Z}$, where $B_c = 15$ for $m^2 = 0.1$ (Winyard, 2016).

Infinite crystal structure proposed to be hexagonal (Hen & Karliner, 2008).

We will show that crystal chunks become the lower energy solution for $B > B_r$ for some $B_r \in \mathbb{Z}$.



(a) $B = 9$ chain.



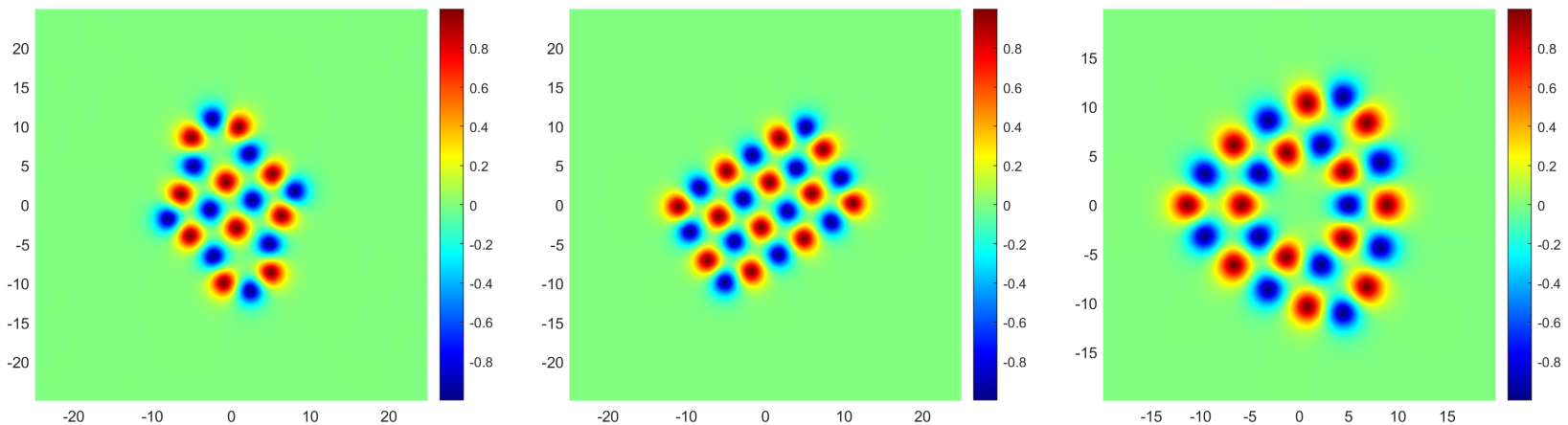
(b) $B = 30$ ring.

Easy plane baby Skyrmions on $\mathbb{R}^2$

For charges $B \leq 6$ with mass $m^2 = 1$, the global minima are $2B$-gons or ring-like solutions (Jäykkä & Speight, 2010).

Chunks of an infinite crystal with a square crystalline structure appear to be the global minima for *almost* all charges $B > 6$.

The easy plane model also exhibits a modular structure with some more exotic local minima consisting of square and polygonal building blocks.



(a) $B = 10$. **(b)** $B = 12$. **(c)** $B = 13$.

Figure 3: φ^1 density plots of local minima in the easy plane model showing an underlying modular structure.

Lattice planar Skyrmions

The physical space of interest is the 2-torus $\Sigma = \mathbb{R}^2/\Lambda$, where Λ is the set of all 2-dimensional period lattices

$$\Lambda = \left\{ \sum_{i=1}^{2} n_i(\alpha X_i) \mid n_i \in \mathbb{Z}, \alpha \in \mathbb{R}^* \right\}$$

and $\{X_1, X_2\}$ is a basis for $\mathbb{R}^2$.

Crystallographic restriction theorem: 5 lattice types in 2-dimensions. Fundamental unit cell is a certain type of a parallelogram.

To find the optimal crystalline structure, we minimize the static energy over all period lattices.

Equivalently, we fix our domain of φ to be $\mathbb{R}^2/\mathbb{Z}^2$ and identify every other torus $\mathbb{R}^2/\Lambda$ with $\mathbb{R}^2/\mathbb{Z}^2$, but with a nonstandard Riemannian metric g. This metric g is the pullback of the usual metric $\bar{g}$ on $\mathbb{R}^2/\Lambda$ via the diffeomorphism $\mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\Lambda$. As we vary Λ then the metric g varies (Speight, 2014).

Lattice planar Skyrmions

Let $F : \mathbb{T}^2 \to \mathbb{R}^2/\Lambda$ be a diffeomorphism with $F \in \mathrm{GL}(2,\mathbb{R})$ and (x_1, x_2) be local coordinates on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.

Identify $\mathrm{GL}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^*$ and let $\mathcal{A} = [X_1\, X_2] \in \mathrm{SL}(2,\mathbb{R})$ and $\alpha \in \mathbb{R}^*$, such that $F = \alpha\mathcal{A}$.

Now identify the Skyrme field as a map $\varphi : \mathbb{T}^2 \to S^2$.

The metric on $\mathbb{T}^2$ is the pullback $g = F^*\bar{g}$ of the flat Euclidean metric $\bar{g}$ on $\mathbb{R}^2/\Lambda$. Explicitly, this is

$$g = \alpha^2 \begin{bmatrix} X_1 \cdot X_1 & X_1 \cdot X_2 \\ X_1 \cdot X_2 & X_2 \cdot X_2 \end{bmatrix}.$$

The Riemannian volume form is simply

$$\mathrm{vol}_g = \sqrt{\det g}\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \alpha^2\, \mathrm{d}x^1 \wedge \mathrm{d}x^2.$$

The Dirichlet energy on $\mathbb{T}^2$ given by

$$\begin{aligned} E_2 &= \frac{1}{2}\int_{\mathbb{T}^2} |\mathrm{d}\varphi|^2 \,\mathrm{vol}_g \\ &= \frac{1}{2}\int_{\mathbb{T}^2} g^{\mu\nu}\partial_\mu\varphi^\alpha\partial_\nu\varphi^\beta h_{\alpha\beta}\sqrt{\det g}\,\mathrm{d}x^1\mathrm{d}x^2 \\ &= \frac{1}{2}\int_{\mathbb{T}^2} \left\{(X_2\cdot X_2)(\partial_1\varphi)^2 - 2(X_2\cdot X_1)(\partial_1\varphi\cdot\partial_2\varphi) + (X_1\cdot X_1)(\partial_2\varphi)^2\right\}\mathrm{d}x^1\,\mathrm{d}x^2. \end{aligned}$$

Likewise, we can compute the Skyrme energy on $\mathbb{T}^2$,

$$\begin{aligned} E_4 &= \frac{\kappa^2}{2}\int_{\mathbb{T}^2} |\varphi^*\omega|^2 \,\mathrm{vol}_g \\ &= \frac{\kappa^2}{4}\int_{\mathbb{T}^2} g^{\alpha\beta}g^{\mu\nu}(\varphi^*\omega)_{\alpha\mu}(\varphi^*\omega)_{\beta\nu}\sqrt{\det g}\,\mathrm{d}x^1\mathrm{d}x^2 \\ &= \frac{\kappa^2}{2\alpha^2}\int_{\mathbb{T}^2} (\partial_1\varphi\times\partial_2\varphi)\cdot(\partial_1\varphi\times\partial_2\varphi)\,\mathrm{d}x^1\,\mathrm{d}x^2 \end{aligned}$$

and the potential energy,

$$E_0 = \alpha^2\int_{\mathbb{T}^2} V[\varphi]\,\mathrm{d}x^1\,\mathrm{d}x^2.$$

Taking the variation of the static energy functional with respect to α,

$$\frac{\partial E}{\partial \alpha} = -\frac{\kappa^2}{\alpha^3} \int_{\mathbb{T}^2} \left(\partial_1 \varphi \times \partial_2 \varphi\right)^2 \mathrm{d}x^1\, \mathrm{d}x^2 + 2\alpha \int_{\mathbb{T}^2} V[\varphi]\, \mathrm{d}x^1\, \mathrm{d}x^2,$$

yields the relation

$$\alpha^2 = \sqrt{\frac{\frac{\kappa^2}{2} \int_{\mathbb{T}^2} \left(\partial_1 \varphi \times \partial_2 \varphi\right)^2 \mathrm{d}x^1\, \mathrm{d}x^2}{\int_{\mathbb{T}^2} V[\varphi]\, \mathrm{d}x^1\, \mathrm{d}x^2}} = \sqrt{\frac{E_4}{E_0}}.$$

Finding the period lattice parameters X_1, X_2 which minimize the Dirichlet energy E_2 is a constrained quadratic optimization problem with the nonlinear constraint $\det([X_1\, X_2]) = 1$.

For notational convenience, let us write

$$\mathcal{E}_{ij} = \int_{\mathbb{T}^2} \left(\partial_i \varphi \cdot \partial_j \varphi\right) \mathrm{d}x^1\, \mathrm{d}x^2.$$

Then the Dirichlet energy E_2 can be expressed in the form

$$E_2 = \frac{1}{2}\boldsymbol{x}^T \mathcal{Q}\boldsymbol{x}, \quad \mathcal{Q} = \begin{bmatrix} \mathcal{E}_{22} & 0 & -\mathcal{E}_{12} & 0 \\ 0 & \mathcal{E}_{22} & 0 & -\mathcal{E}_{12} \\ -\mathcal{E}_{12} & 0 & \mathcal{E}_{11} & 0 \\ 0 & -\mathcal{E}_{12} & 0 & \mathcal{E}_{11} \end{bmatrix},$$

where $\boldsymbol{x} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$ is a 4-vector and $\mathcal{Q}$ a 4×4-symmetric matrix.

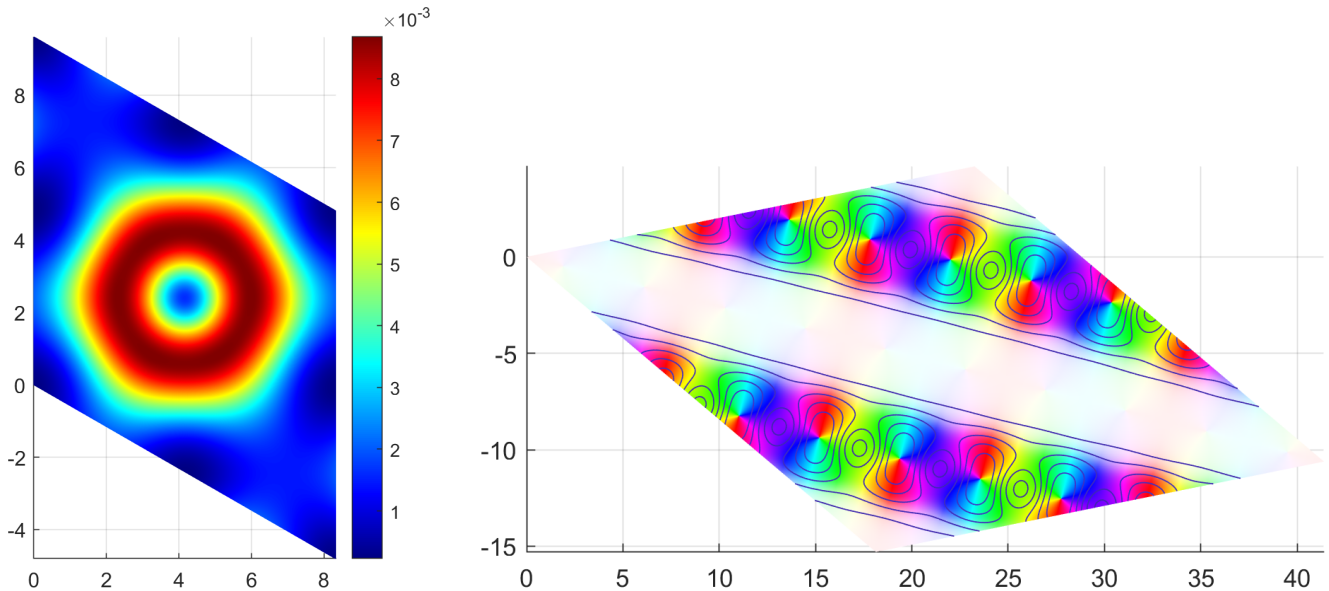
This constrained quadratic optimization problem can be solved by including the Lagrange term $\gamma(\det([X_1\, X_2]) - 1)$, where $\gamma \in \mathbb{R}^*$ is a Lagrange multiplier. This reduces the problem to an eigenvalue problem

$$\mathcal{B}\boldsymbol{x} = \gamma\boldsymbol{x}, \quad \mathcal{B} = \begin{bmatrix} 0 & \mathcal{E}_{12} & 0 & -\mathcal{E}_{11} \\ -\mathcal{E}_{12} & 0 & \mathcal{E}_{11} & 0 \\ 0 & \mathcal{E}_{22} & 0 & -\mathcal{E}_{12} \\ -\mathcal{E}_{22} & 0 & \mathcal{E}_{12} & 0 \end{bmatrix}.$$

Standard crystalline structure

The optimal lattice is found to be an equianharmonic lattice with a hexagonal crystalline structure.

The infinite crystal has energy $\mathcal{E}_{\text{crystal}} = 1.4543$, which is lower than the infinite single chain energy $\mathcal{E}_{\text{chain}} = 1.4548$ and infinite adjacent chains energy $\mathcal{E}_{\text{2-chains}} = 1.4545$.



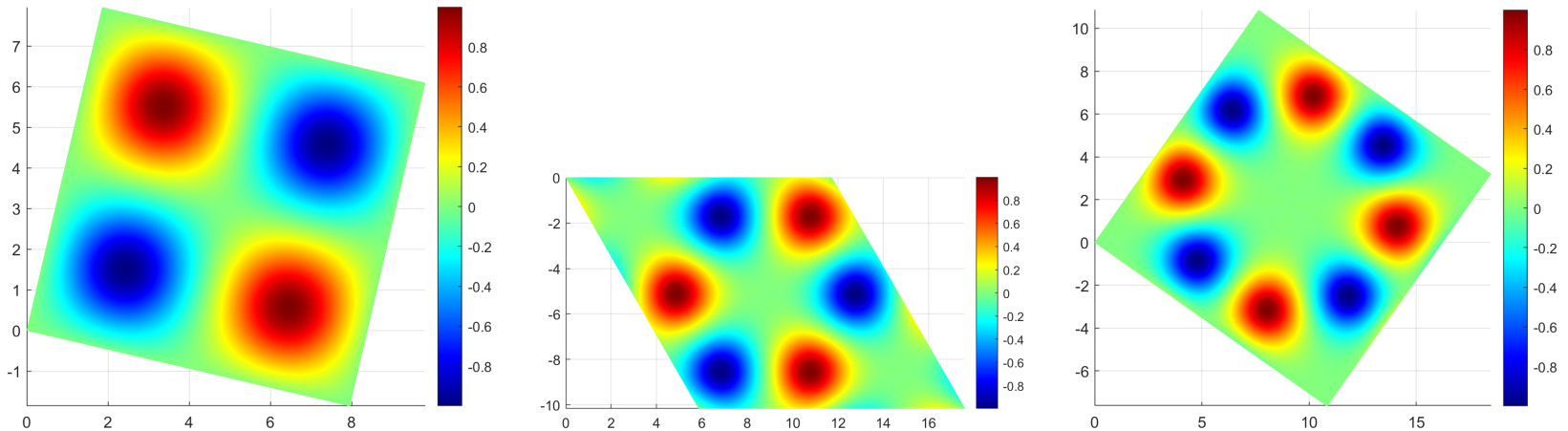
(a) **(b)**

Figure 4: Energy density plots of (a) minimal energy hexagonal crystal structure and (b) adjacent chain crystal structure in their corresponding optimal lattices.

Easy plane crystalline structure

The lowest energy crystal structure is a square of half lumps in a square lattice for $B = 2$, with energy $\mathcal{E}_{B=2} = 1.5152$.

Two other crystalline structures were found with slightly higher energies: a hexagonal crystalline structure in an equianharmonic lattice for $B = 3$ with energy $\mathcal{E}_{B=3} = 1.5207$, and an octogonal crystal structure in a square lattice with energy $\mathcal{E}_{B=4} = 1.5228$.



(a) Square crystal. **(b)** Hexagonal crystal. **(c)** Octogonal crystal.

Figure 5: φ^1 density plots of the (a) $B = 2$ square crystal, (b) $B = 3$ hexagonal crystal and (c) $B = 4$ octogonal crystal in their corresponding optimal lattices.

Crystal slab model on $\mathbb{R} \times S^1$

Physical space is the cylinder $\Sigma = \mathbb{R} \times S^1$.

This corresponds to a Dirichlet boundary condition in the x_2-direction, $\lim_{|x_2| \to \infty} = \varphi_\infty$, and a periodic boundary condition in the x_1-direction, $\varphi(x_1, x_2) = \varphi(x_1 + n_1 L, x_2)$, where $n_1 \in \mathbb{Z}$.

Planar Skyrmions are layered on an infinite cylinder of width $L = L_{\text{crystal}}$, in the optimal crystal arrangement, to estimate the surface energy per unit length.

Applying a least squares fit to the model

$$\mathcal{E}_{\text{slab}} = \mathcal{E}_{\text{crystal}} + 2\frac{L_{\text{crystal}}}{2n}\mathcal{E}_{\text{surf}},$$

where $\mathcal{E}_{\text{surf}}$ is the surface energy per unit length and number of layers n, we can calculate $\mathcal{E}_{\text{surf}}$.

For the standard potential we find that $\mathcal{E}_{\text{surf}} = 6.58 \times 10^{-4}$, and for the easy plane potential $\mathcal{E}_{\text{surf}} = 0.0025$.

Crystal slab layering

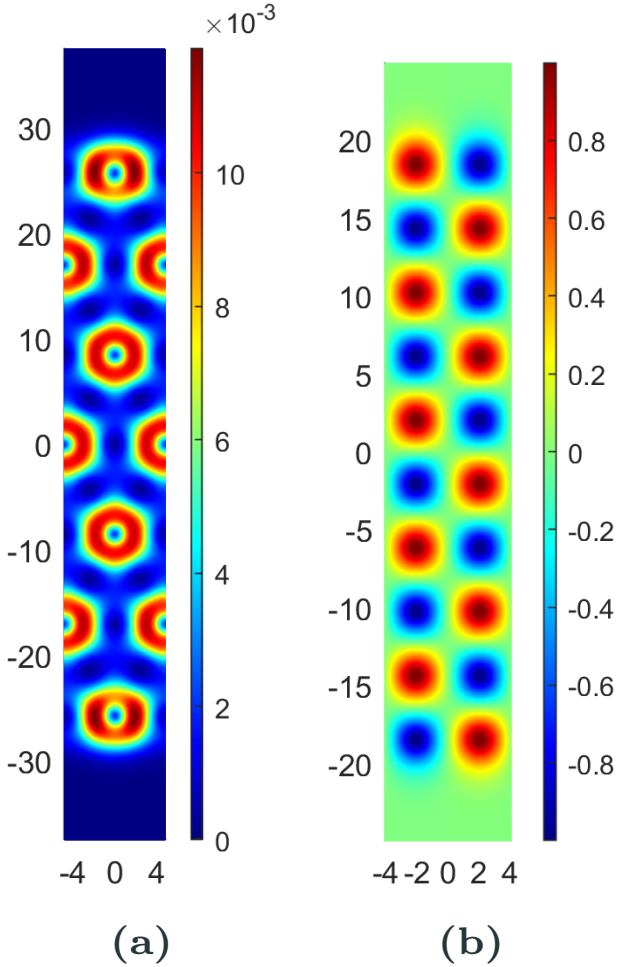


Figure 6: Energy density and φ^1 density plots showing a (a) 7-layer standard hexagonal crystal slab and (b) a 5-layer easy plane square crystal slab.

Crystal chunk approximation

Want to find the crystal chunk that minimises its boundary length and hence its surface energy contribution $\Rightarrow$ isoperimetric problem.

Isoperimetric inequality in $\mathbb{R}^2$ is $L^2 \geq 4\pi A$ (Osserman, 1978), with equality iff the crystal boundary is a cirle.

Minimal crystal surface energy $\Rightarrow L^2 = 4\pi A$ (crystal disks).

Using this assumption, we can express chunks of the crystal in the form

$$\mathcal{E}_{\text{chunk}} = \mathcal{E}_{\text{crystal}} + 2\mathcal{E}_{\text{surf}}\sqrt{\frac{\pi}{B\rho_B}},$$

where ρ_B is the charge per unit area.

Can empirically compare chains, rings and crystal chunks using ring and chain approximations obtained by (Winyard, 2016).

Crystal chunk solutions become global minima for approximately $B > 877$.

Rings, chains and chunks

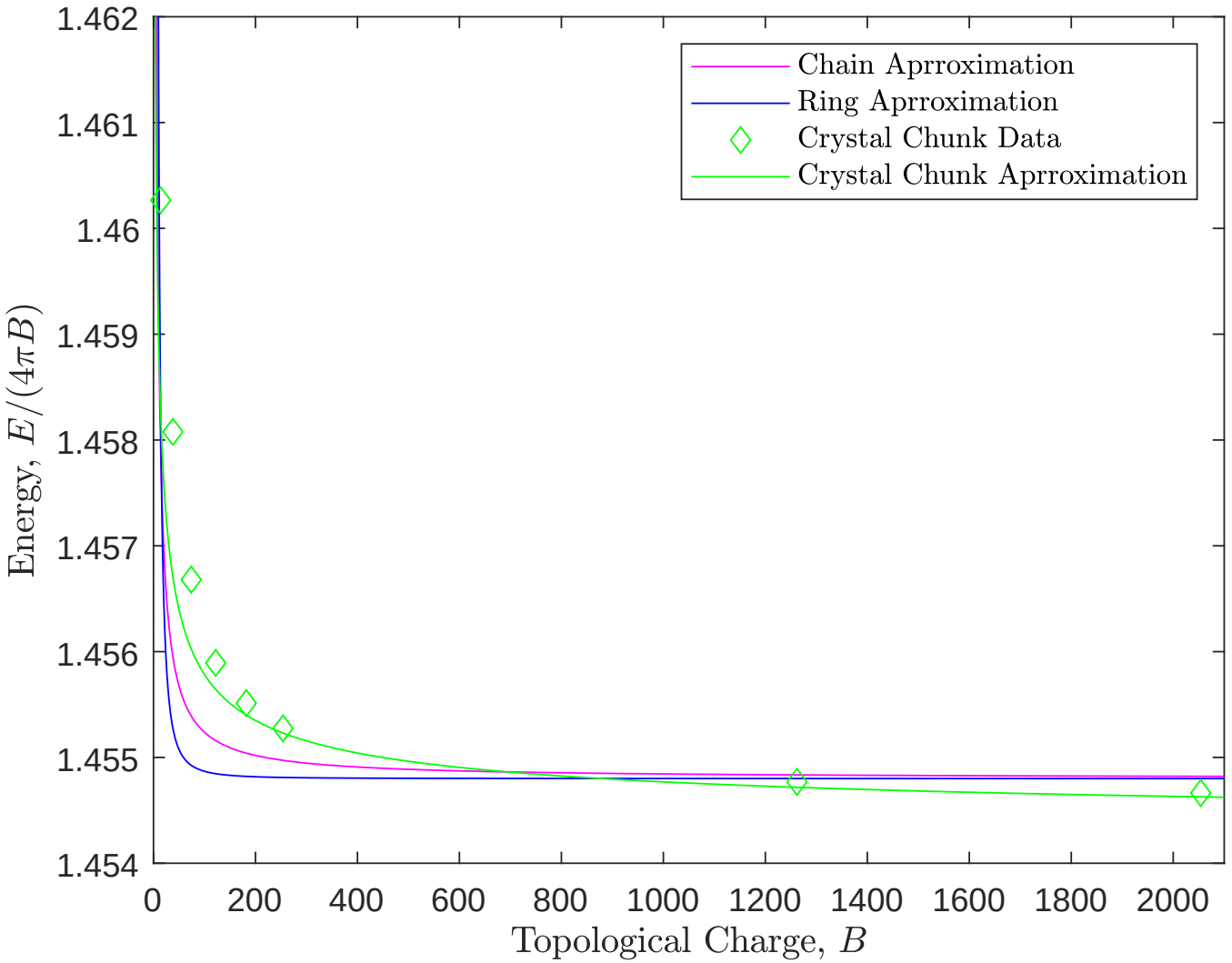
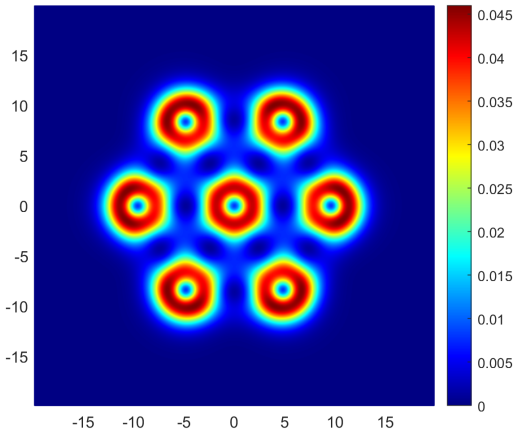
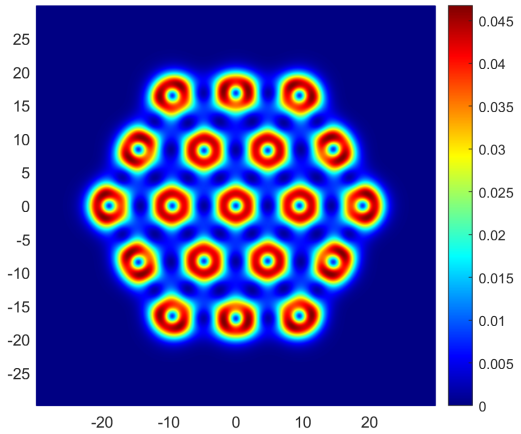


Figure 7: Comparison of ring, chain and crystal chunk approximations.

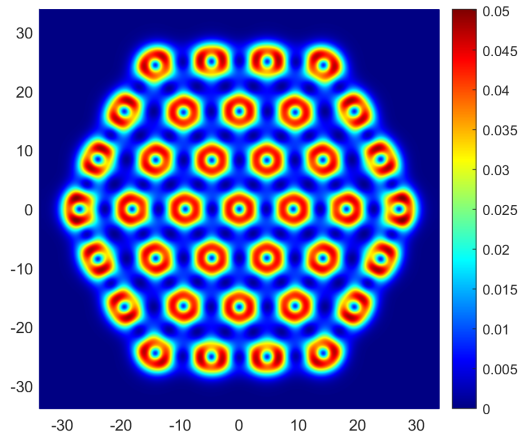
Standard crystal chunks



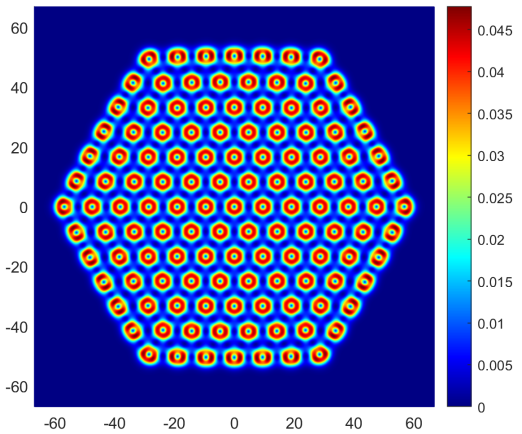
(a) $n = 3, B = 14$.



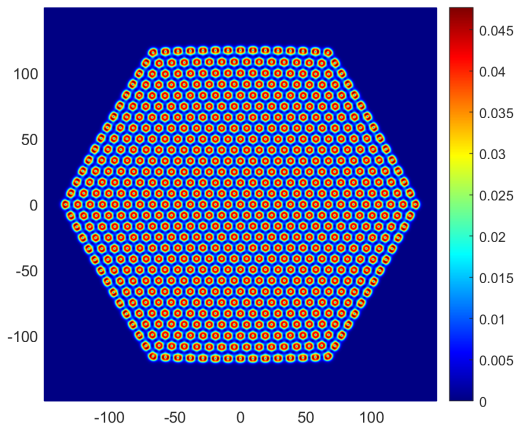
(b) $n = 5, B = 38$.



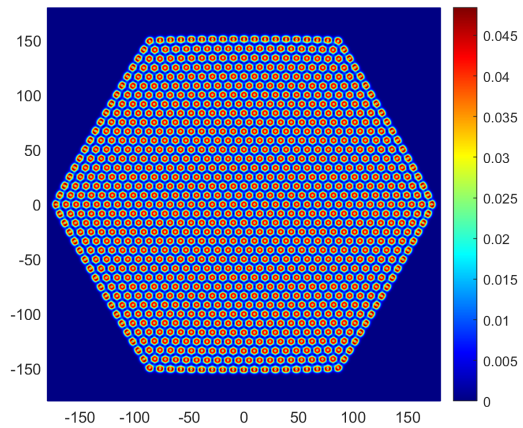
(c) $n = 7, B = 74$.



(d) $n = 13, B = 254$.



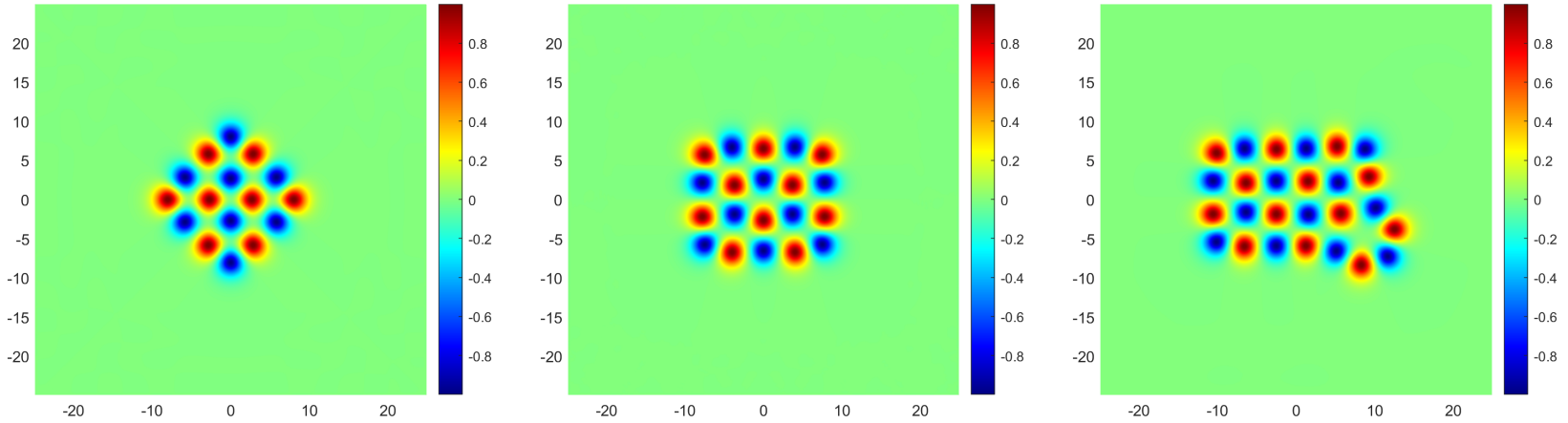
(e) $n = 29, B = 1262$.



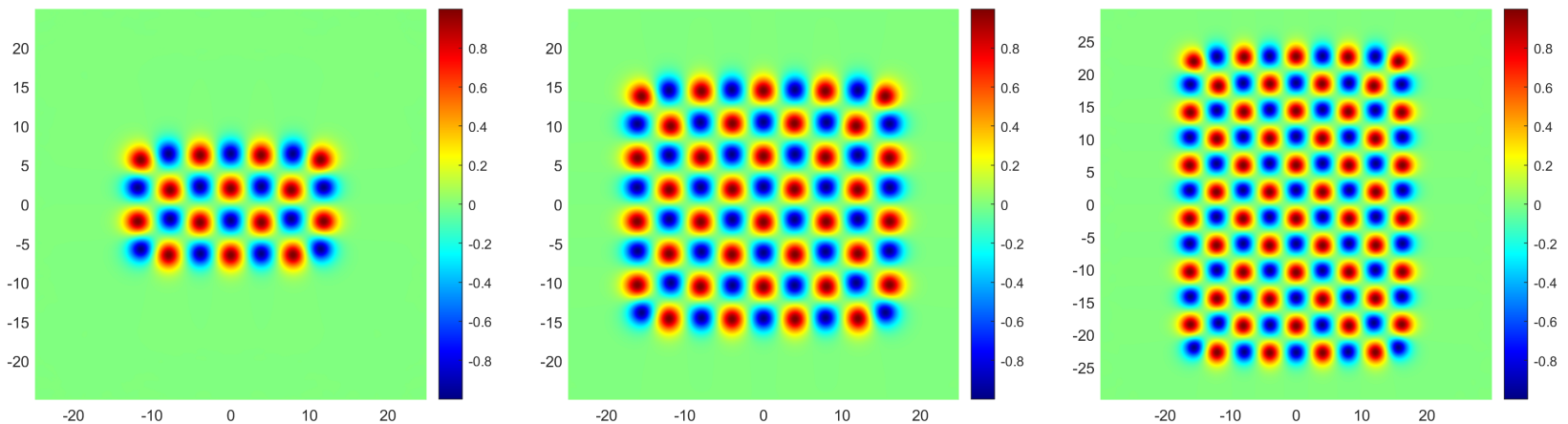
(f) $n = 37, B = 2054$.

Figure 8: Energy density plots of crystal chunk solutions in the standard model, n is the equivalent crystal slab thickness.

Easy plane crystal chunks



(a) $B = 8$. (b) $B = 10$. (c) $B = 13$.



(d) $B = 14$. (e) $B = 36$. (f) $B = 54$.

Figure 9: φ^1 density plots of crystal chunk solutions in the easy plane model. These are all global minima.

Summary

Optimal crystalline structure is hexagonal for the standard potential, and square for the easy plane potential.

Crystal chunks are global minima for *low* charges B for the easy plane model, whereas they are for *very large* charges B for the standard model.

Optimal crystal structure in the Skyrme model is thought to be cube of half Skyrmions (Kugler & Shtrikman, 1988).

Generalising this method, will this half Skyrmion cube structure prevail?

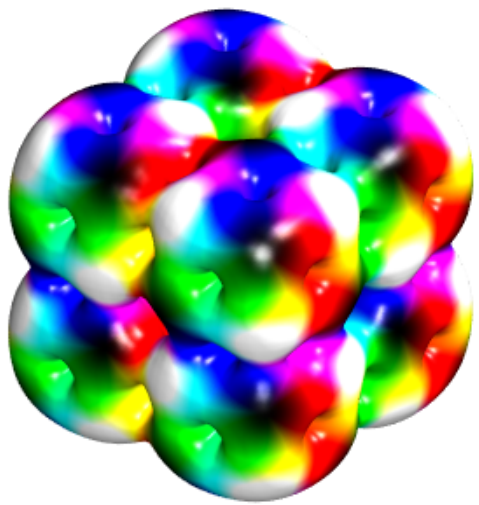


Figure 10: $B = 32$ crystal chunk solution in the Skyrme model. Runge coloring scheme as detailed in (Feist *et al.*, 2013).