Numerical Solutions for Skyrme Models

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Outline

Mathematical framework for Skyrmions

The Skyrme model

Initial configurations

- Hedgehog Ansatz
- Rational map ansatz

Arrested Newton flow

Skyrme–Faddeev model

Mathematical framework for Skyrmions

- The Skyrme model's natural setting is Riemannian geometry.
- Let (M, g_M) and (N, g_N) be 3-dimensional, orientable and connected Riemannian manifolds.
- A Skyrmion is a topologically non-trivial map $U : M \to N$.
- The energy functional will be a measure of the extent to which the map U is metric preserving.
- Skyrmions are field configurations that are minimal energy solutions to the associated Euler–Lagrange equations.

Mathematical framework for Skyrmions

- Introduce coordinates p^i on M and U^j on N and orthonormal frame fields $\boldsymbol{m}_\alpha$ on M and $\boldsymbol{n}_\beta$ on N.
- Represent U by the functions $U^j(p^1, p^2, p^3)$.
- The deformation induced by U at p can be determined by the Jacobian matrix

$$J_{\alpha\beta} = m_\alpha^i \frac{\partial U^j}{\partial p^i} n_{\beta j}.$$

- Geometric distortion is unaffected by rotations in M and isorotations in N.
- In analogy with elasticity theory, we can express the energy of the Skyrme field U in terms of its strain tensor D.

Basic invariants of the strain tensor D

- The strain tensor is defined as $D = JJ^T$.
- This is a 3×3 symmetric, positive-definite matrix with eigenvalues $\lambda_1^2, \lambda_2^2, \lambda_3^2$.
- Basic invariants of D are the coefficients in the characteristic polynomial $\chi_D(g_M) = \det(D - g_M \mathrm{id})$. These are

$$\begin{aligned} \operatorname{Tr} D &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2, \\ \frac{1}{2}(\operatorname{Tr} D)^2 - \frac{1}{2}\operatorname{Tr} D^2 &= \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2, \\ \det D &= \lambda_1^2\lambda_2^2\lambda_3^2. \end{aligned}$$

- The deviation of the eigenvalues of D from unity determines the deformation induced by U. If $D = \mathrm{id}$, then there is no deformation and U is locally an isometry.

Constructing invariants from the strain tensor D

- The simplest invariant is the Dirichlet energy

$$\begin{aligned} E_2 &= \int_M \operatorname{Tr} D \sqrt{\det g_M}\, \mathrm{d}^3x \\ &= \int_M \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right) \sqrt{\det g_M}\, \mathrm{d}^3x, \end{aligned}$$

whose critical points are harmonic functions.

- The second invariant is the Skyrme energy

$$\begin{aligned} E_4 &= \int_M \left(\frac{1}{2}(\operatorname{Tr} D)^2 - \frac{1}{2}\operatorname{Tr} D^2\right) \sqrt{\det g_M}\, \mathrm{d}^3x \\ &= \int_M \left(\lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2\right) \sqrt{\det g_M}\, \mathrm{d}^3x. \end{aligned}$$

Constructing invariants from the strain tensor D

- In the usual Skyrme model, the static energy functional is constructed from both of these invariants:

$$\begin{aligned} E &= \int_M \left(\operatorname{Tr} D + \frac{1}{2} (\operatorname{Tr} D)^2 - \frac{1}{2} \operatorname{Tr} D^2 \right) \sqrt{\det g_M} \, \mathrm{d}^3 x \\ &= \int_M \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_1^2 \lambda_2^2 + \lambda_2^2 \lambda_3^2 + \lambda_3^2 \lambda_1^2 \right) \sqrt{\det g_M} \, \mathrm{d}^3 x. \end{aligned}$$

- The third basic invariant actually gives us the topological degree of the map U,

$$\begin{aligned} B &= \frac{1}{2\pi^2} \int_M \sqrt{\det D} \sqrt{\det g_M} \, \mathrm{d}^3 x \\ &= \frac{1}{2\pi^2} \int_M \lambda_1 \lambda_2 \lambda_3 \sqrt{\det g_M} \, \mathrm{d}^3 x. \end{aligned}$$

Faddeev–Bogomolny energy bound

- We can construct a lower bound on the energy of a Skyrmion within a given homotopy class by writing the energy E in the form

$$E = \int_M \big((\lambda_1 \pm \lambda_2\lambda_3)^2 + (\lambda_2 \pm \lambda_3\lambda_1)^2 + (\lambda_3 \pm \lambda_1\lambda_2)^2 \\ +6\lambda_1\lambda_2\lambda_3\big) \sqrt{\det g_M}\, \mathrm{d}^3x,$$

  and using the simple inequality

$$(\lambda_1 \pm \lambda_2\lambda_3)^2 + (\lambda_2 \pm \lambda_3\lambda_1)^2 + (\lambda_3 \pm \lambda_1\lambda_2)^2 \geq 0.$$

- This gives us the Faddeev–Bogomolny lower bound on the energy,

$$E \geq 12\pi^2 |B|.$$

The Skyrme model

- In the usual Skyrme model, physical space is $M = \mathbb{R}^3$ and the target is $N = \mathrm{SU}(2)$.
- The static Skyrme model consists of a single scalar field, $U : \mathbb{R}^3 \to \mathrm{SU}(2)$ given explicitly by

$$U(\mathbf{x}) = \sigma(\mathbf{x})\mathrm{id} + i\boldsymbol{\pi}(\mathbf{x}) \cdot \boldsymbol{\tau},$$

  where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ are the pion fields, $\boldsymbol{\tau}$ are the Pauli matrices and σ is the sigma field.
- Nuclei are modelled as topological solitons in a nonlinear field theory of pions.
- Realized as a low energy effective field theory for QCD in the large colour limit, in which nuclei become classical and heavy.

The Skyrme model

- The strain tensor associated to $U : \mathbb{R}^3 \to \mathrm{SU}(2)$ is

$$D_{ij} = -\frac{1}{2}\,\mathrm{Tr}(L_i L_j),$$

  where the $\mathfrak{su}(2)$-valued (left) current $L_i = U^\dagger(\partial_i U)$ is the pullback of the Maurer–Cartan 1-form on $\mathrm{SU}(2)$.
- The corresponding static energy functional is

$$E = \int_{\mathbb{R}^3} \left\{ -\frac{1}{2}\,\mathrm{Tr}(L_i L_i) - \frac{1}{16}\,\mathrm{Tr}\left([L_i, L_j][L_i, L_j]\right) + V(U) \right\} \mathrm{d}^3 x,$$

  where $V(U) = m^2\,\mathrm{Tr}(\mathrm{id} - U)$ is the usual pion mass potential.
- The energy E is invariant under translations and rotations in physical space, as well as rotations in isospace, $U \to AUA^\dagger$ for $A \in \mathrm{SU}(2)$.

The Skyrme model

- We have the nonlinear σ-model constraint $\sigma^2 + \boldsymbol{\pi} \cdot \boldsymbol{\pi} = 1$ and the identification $\mathrm{SU}(2) \cong S^3$.
- Finite energy configurations require us to impose the boundary condition $U \to \mathrm{id}$ as $|\mathbf{x}| \to \infty$. This gives us the one-point compactification of space $\mathbb{R}^3 \cup \{\infty\} \cong S^3$.
- Topologically, the Skyrme field is a map $U : S^3 \to S^3$, with winding number $B \in \pi_3(S^3) \cong \mathbb{Z}$.
- The topological charge B is identified with the physical baryon number, with explicit integral form

$$B = -\frac{1}{24\pi^2} \int_{\mathbb{R}^3} \epsilon_{ijk} \, \mathrm{Tr}(L_i L_j L_k) \, \mathrm{d}^3 x.$$

- The minimum energy field configurations for each baryon number B are known as Skyrmions and their energy E is identified with their rest mass.

Derrick's non-existence theorem

- Apply a rescaling of the spatial coordinates $\mathbf{x} \mapsto \lambda \mathbf{x}$.
- The energy terms becomes $e_2 = \frac{E_2}{\lambda}$, $e_4 = \lambda E_4$ and $e_0 = \frac{E_0}{\lambda^3}$.
- Rescaled energy is simply

$$e(\lambda) = \frac{E_2}{\lambda} + \lambda E_4 + \frac{E_0}{\lambda^3}.$$

- λ must minimise $e(\lambda)$ at $\lambda = 1$, which requires $E_4 = E_2 + 3E_0$.
- The terms E_2 & E_0 and E_4 scale in opposite ways, so Skyrmions cannot:
    1. expand to cover all of space,
    2. contract to a localized single point.

- For numerics, it is convenient to write the Skyrme field as a 4-vector $U = \phi_\mu e_\mu$, where $\phi_\mu = (\sigma, \pi_j)$ and $e_\mu = (\mathrm{id}, i\boldsymbol{\tau}_j)$, such that $\phi_\mu \phi_\mu = 1$.
- Substituting this into the Skyrme Lagrangian we obtain a non-linear σ-model form,

$$L = \int_{\mathbb{R}^3} \Big\{ \partial_i \boldsymbol{\phi} \cdot \partial^i \boldsymbol{\phi} - \frac{1}{2} \left(\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi} \right) \cdot \left(\partial^i \boldsymbol{\phi} \times \partial^j \boldsymbol{\phi} \right) - 2m^2 (1 - \sigma) \Big\} \, \mathrm{d}^3 x.$$

- Then the static energy functional is defined by

$$E = \int_{\mathbb{R}^3} \left\{ \partial_i \boldsymbol{\phi} \cdot \partial_i \boldsymbol{\phi} + \frac{1}{2} \left(\partial_i \boldsymbol{\phi} \times \partial_j \boldsymbol{\phi} \right)^2 + 2m^2 (1 - \sigma) \right\} \mathrm{d}^3 x.$$

How to construct Skyrmions

- The basic $B = 1$ Skyrmion is based on a spherical hedgehog ansatz where the radial profile function $f(r)$ is the solution of an ODE.
- To construct higher charge Skyrmions there are a few methods:
  - One is to place $B = 1$ Skyrmions orientated in the attractive channel. The best arrangements are on a subcluster of a bravais lattice, the most favourable being a face-centred cubic (FCC) lattice.
  - Another is to build Skyrme fields using a rational map approximation $S^2 \to S^2$ and a radial profile $f(r)$.
  - One could also relate Skyrmions to instantons or monopoles.

Ansatz

- The hedgehog field is $U(\mathbf{x}) = \exp\left(if(r)\hat{\mathbf{x}} \cdot \boldsymbol{\tau}\right)$. In terms of the $\boldsymbol{\pi}$ and σ fields,

$$\sigma = \cos f(r), \quad \boldsymbol{\pi} = \sin f(r)\hat{\mathbf{x}}.$$

  These are known as hedgehogs because the pion fields point radially outwards.
- The profile function $f(r)$ must satisfy the boundary conditions $f(\infty) = 0$ and $f(0) = n\pi$ with $n \in \mathbb{Z}$.
- Such a hedgehog solution has baryon number $B = n$ since

$$B = -\frac{1}{2\pi^2} \int_0^\infty \frac{\sin^2 f}{r^2} \frac{\mathrm{d}f}{\mathrm{d}r} 4\pi r^2 dr = \frac{1}{\pi} f(0) = n.$$

Ansatz

- For the hedgehog ansatz, the (massless) static energy is

$$E = 4\pi \int_0^\infty \left[r^2 \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 + 2\sin^2 f \left(1 + \left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 \right) + \frac{\sin^4 f}{r^2} \right] \mathrm{d}r.$$

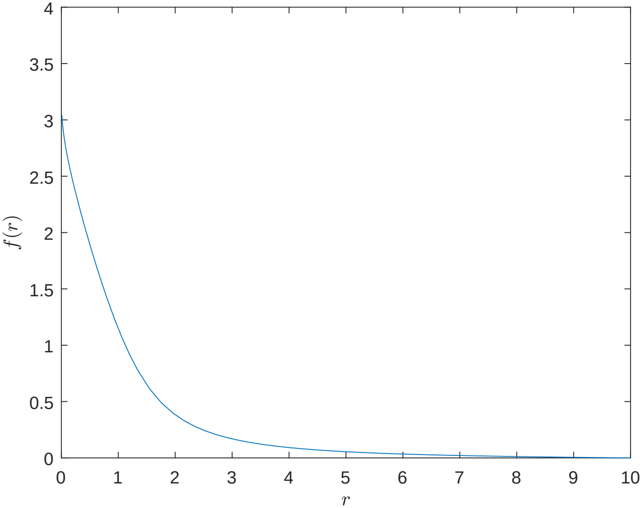
- The corresponding field equations reduce to the second order non-linear ordinary differential equation

$$\left(r^2 + 2\sin^2 f \right) \frac{\mathrm{d}^2 f}{\mathrm{d}r^2} + 2r \frac{\mathrm{d}f}{\mathrm{d}r} + \sin 2f \left[\left(\frac{\mathrm{d}f}{\mathrm{d}r} \right)^2 - 1 - \frac{\sin^2 f}{r^2} \right] = 0.$$

- This ODE can only be solved numerically and has a unique solution for each B.
- The $B = 1$ Skyrmions has energy $E_1 = 1.232 \times 12\pi^2$, which is greater than the Bogomolny bound, $E_1 > 12\pi^2$.

Ansatz

- Using a shooting method, the profile function $f(r)$ for the $B = 1$ hedgehog ansatz is found to be:



$B = 1$ Skyrmion

- The $B = 1$ Skyrmion is visualised by plotting a constant baryon density isosurface, e.g. $B = 0.2$, and is coloured using the Runge colour sphere.

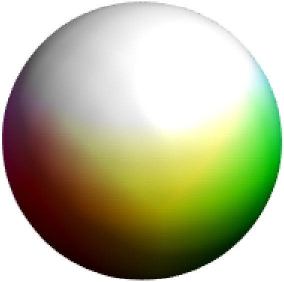


Figure 1: Constant baryon density isosurface plot of the minimal energy Skyrmion with massive pions for baryon number $B = 1$.

Rational map ansatz

- Rational maps are functions from $S^2 \to S^2$, whereas as the Skyrme field is a map $\mathbb{R}^3 \to S^3$.
- Identify the rational map target S^2 with spheres of constant latitude on S^3, and the rational map domain S^2 with spheres in $\mathbb{R}^3$ of radius r.
- Using polar coordinates for $\mathbb{R}^3$, $z = \tan(\theta/2)\exp(i\varphi)$, with radius r, the rational map ansatz is

$$U(r,z) = \exp\left[\frac{if(r)}{1+|R|^2}\begin{pmatrix} 1-|R|^2 & 2\bar{R} \\ 2R & |R|^2-1 \end{pmatrix}\right],$$

where $f(r)$ is a radial profile function with $f(0)=\pi$ and $f(\infty)=0$, and $R(z)=p(z)/q(z)$ is a rational map of degree $B=\max(\deg p, \deg q)$.

Rational map ansatz

- Substituting the rational map ansatz into the static energy functional yields

$$E = 4\pi \int_0^\infty r^2 \left\{ \left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 2B\sin^2 f \left(\left(\frac{\mathrm{d}f}{\mathrm{d}r}\right)^2 + 1 \right) + \mathcal{I}\frac{\sin^4 f}{r^2} + 2m^2(\cos f - 1) \right\} \mathrm{d}r,$$

  where $\mathcal{I}$ is the purely angular integral to be minimised for choice of rational map R:

$$\mathcal{I} = \frac{1}{4\pi} \int \left(\frac{1+|z|^2}{1+|R|^2} \left| \frac{\mathrm{d}R}{\mathrm{d}z} \right| \right)^4 \frac{2i\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}.$$

- Optimising $\mathcal{I}$ and the profile function $f(r)$ gives approximate Skyrmions, but further numerical relaxation is required to find true Skyrmions.

Rational map ansatz

- Below are some of the well known rational maps of high symmetry which we use as initial configurations for the numerical relaxation:

B	$R(z)$	Symmetry
1	$R(z) = z$	$O(3)$
2	$R(z) = z^2$	$O(2) \times \mathbb{Z}$
3	$R(z) = \frac{z^3 - \sqrt{3}iz}{\sqrt{3}iz^2 - 1}$	T_d
4	$R(z) = \frac{z^4 + 2\sqrt{3}iz^2 + 1}{z^4 - 2\sqrt{3}iz^2 + 1}$	O_h
5	$R(z) = \frac{z(z^4 + bz^2 + a)}{az^4 - bz^2 + 1}$	D_{2d}
6	$R(z) = \frac{z^4 - a}{z(az^4 + 1)}$	D_{4d}
7	$R(z) = \frac{bz^6 - 7z^4 - bz^2 - 1}{z(z^6 + bz^4 + 7z^2 - b)}$	Y_h

- The Skyrmions are visualised by plotting a constant baryon density isosurface, e.g. $B = 0.2$, and are coloured using the Runge colour sphere.

Arrested Newton flow

- To find minima of the static energy, we must numerically relax the Skyrme field.
- The numerical methods are carried out on a $N_1 \times N_2 \times N_3$ grid with lattice spacings $\Delta x_1, \Delta x_2, \Delta x_3$.
- The Skyrme energy is then discretised using a 4th order, 5-point stencil, central finite-difference scheme.
- The first order and second order spatial derivatives with respect to the local coordinate x_1, with the other derivatives defined analogously, are given respectively by

$$\frac{\partial \phi_{a_{i,j,k}}}{\partial x_1} = \frac{\frac{1}{12}\phi_{a_{i-2,j,k}} - \frac{2}{3}\phi_{a_{i-1,j,k}} + \frac{2}{3}\phi_{a_{i+1,j,k}} - \frac{1}{12}\phi_{a_{i+2,j,k}}}{\Delta x_1},$$

$$\frac{\partial^2 \phi_{a_{i,j,k}}}{\partial x_1^2} = \frac{-\frac{1}{12}\phi_{a_{i-2,j,k}} + \frac{4}{3}\phi_{a_{i-1,j,k}} - \frac{5}{2}\phi_{a_{i,j,k}} + \frac{4}{3}\phi_{a_{i+1,j,k}} - \frac{1}{12}\phi_{a_{i+2,j,k}}}{(\Delta x_1)^2}.$$

Arrested Newton flow

- This yields a discrete approximation $E_{\text{dis}}[\boldsymbol{\phi}]$ to the static energy functional $E[\boldsymbol{\phi}]$, which we can regard as a function $E_{\text{dis}} : \mathcal{C} \to \mathbb{R}$.
- The discretised configuration space is the manifold $\mathcal{C} = (S^3)^{N_1\, N_2\, N_3} \subset \mathbb{R}^{4\, N_1\, N_2\, N_3}$.
- We solve Newton's equations of motion for a particle on the discretised configuration space $\mathcal{C}$ with potential energy E_{dis}.
- Explicitly, we are solving the system of 2nd order ODEs

$$\ddot{\boldsymbol{\phi}}(t) = -\frac{\delta E_{\text{dis}}}{\delta \boldsymbol{\phi}}[\boldsymbol{\phi}], \quad \boldsymbol{\phi}(0) = \boldsymbol{\phi}_0, \tag{1}$$

with initial velocity $\dot{\boldsymbol{\phi}}(0) = \mathbf{0}$.

- Setting $\boldsymbol{\psi}(t) = \dot{\boldsymbol{\phi}}$ as the velocity with $\boldsymbol{\psi}(0) = \dot{\boldsymbol{\phi}}(0) = \mathbf{0}$ reduces the problem to the coupled system of 1st order ODEs

$$\frac{d}{dt}\begin{bmatrix}\boldsymbol{\phi} \\ \boldsymbol{\psi}\end{bmatrix} = \begin{bmatrix}\boldsymbol{\psi} \\ -\frac{\delta E_{\text{dis}}}{\delta \boldsymbol{\phi}}[\boldsymbol{\phi}]\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{\phi}(0) \\ \boldsymbol{\psi}(0)\end{bmatrix} = \begin{bmatrix}\boldsymbol{\phi}_0 \\ 0\end{bmatrix}.$$

- We implement a 4th order Runge–Kutta method to solve this coupled system.
- Then the evolution of the velocity $\boldsymbol{\psi}$ is given by

$$\boldsymbol{\psi}_{n+1} = \boldsymbol{\psi}_n + \frac{1}{6}\left((\boldsymbol{l}_1)_n + 2(\boldsymbol{l}_2)_n + 2(\boldsymbol{l}_3)_n + (\boldsymbol{l}_4)_n\right),$$

and the Skyrme field $\boldsymbol{\phi}$ evolution is given by

$$\boldsymbol{\phi}_{n+1} = \boldsymbol{\phi}_n + \frac{1}{6}\left((\boldsymbol{k}_1)_n + 2(\boldsymbol{k}_2)_n + 2(\boldsymbol{k}_3)_n + (\boldsymbol{k}_4)_n\right).$$

Arrested Newton flow

- The slopes $\boldsymbol{k}_i, \boldsymbol{l}_i$ for $1 \leq i \leq 4$, with time-step h, for the Skyrme field $\boldsymbol{\phi}_n$ and the velocity $\boldsymbol{\psi}_n$ are

$$
\begin{aligned}
(\boldsymbol{k}_1)_n &= h\,\boldsymbol{\psi}_n, & (\boldsymbol{l}_1)_n &= -h\,\frac{\delta E_{\mathrm{dis}}}{\delta \boldsymbol{\phi}}\left[\boldsymbol{\phi}_n\right],\\
(\boldsymbol{k}_2)_n &= h\left(\boldsymbol{\psi}_n + \frac{1}{2}(\boldsymbol{l}_1)_n\right), & (\boldsymbol{l}_2)_n &= -h\,\frac{\delta E_{\mathrm{dis}}}{\delta \boldsymbol{\phi}}\left[\boldsymbol{\phi}_n + \frac{1}{2}(\boldsymbol{k}_1)_n\right],\\
(\boldsymbol{k}_3)_n &= h\left(\boldsymbol{\psi}_n + \frac{1}{2}(\boldsymbol{l}_2)_n\right), & (\boldsymbol{l}_3)_n &= -h\,\frac{\delta E_{\mathrm{dis}}}{\delta \boldsymbol{\phi}}\left[\boldsymbol{\phi}_n + \frac{1}{2}(\boldsymbol{k}_2)_n\right],\\
(\boldsymbol{k}_4)_n &= h\left(\boldsymbol{\psi}_n + (\boldsymbol{l}_3)_n\right), & (\boldsymbol{l}_4)_n &= -h\,\frac{\delta E_{\mathrm{dis}}}{\delta \boldsymbol{\phi}}\left[\boldsymbol{\phi}_n + (\boldsymbol{k}_3)_n\right].
\end{aligned}
$$

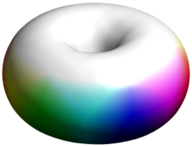
- Main advantage of ANF: the field will accelerate towards an energy minimum.
- So that the field does not overshoot, we take out all the kinetic energy whenever the potential energy is increasing.

Arrested Newton flow

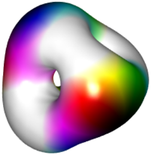
- The flow then terminates when every component of the energy gradient $\frac{\delta E_{\text{dis}}}{\delta \boldsymbol{\phi}}$ is zero to within a pre-assigned tolerance.
- It is essential that we enforce the constraint $\boldsymbol{\phi} \cdot \boldsymbol{\phi} = 1$.
- Normally done by including a Lagrange multiplier term into the Lagrangian.
- To do this numerically we have to pull our target space back onto S^3. This is done by normalizing the Skyrme field $\boldsymbol{\phi}$ each loop, $\phi_a \to \frac{\phi_a}{\sqrt{\boldsymbol{\phi} \cdot \boldsymbol{\phi}}}$.
- We also need to project out the component of the energy gradient, and velocity, in the direction of Skyrme field,

$$\frac{\delta \mathcal{E}}{\delta \phi_a} \to \frac{\delta \mathcal{E}}{\delta \phi_a} - \left(\frac{\delta \mathcal{E}}{\delta \boldsymbol{\phi}} \cdot \boldsymbol{\phi} \right) \frac{\phi_a}{\sqrt{\boldsymbol{\phi} \cdot \boldsymbol{\phi}}},$$

$$\psi_a \to \psi_a - (\boldsymbol{\psi} \cdot \boldsymbol{\phi}) \frac{\phi_a}{\sqrt{\boldsymbol{\phi} \cdot \boldsymbol{\phi}}}.$$

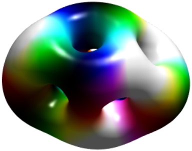


(a) $B = 2 : O(2)$

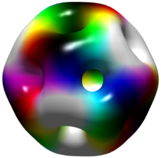


(b) $B = 3 : T_d$

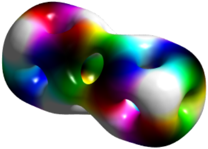
(c) $B = 4 : O_h$



(d) $B = 5 : D_{2d}$



(e) $B = 7 : Y_h$



(f) $B = 8 : D_{4h}$

Table 1: Constant baryon density isosurface plots of the minimal energy Skyrmions with massive pions for baryon numbers $B \leq 8$.

Skyrme–Faddeev model

- The Skyrme–Faddeev model involves a map $\boldsymbol{\phi} : \mathbb{R}^3 \to S^2$, which is realized as a 3-vector $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$.
- The static energy of the model is given by
$$E = \frac{1}{32\pi^2\sqrt{2}} \int_{\mathbb{R}^3} \left\{ \partial_i\boldsymbol{\phi} \cdot \partial_i\boldsymbol{\phi} + \frac{1}{2} \left(\partial_i\boldsymbol{\phi} \times \partial_j\boldsymbol{\phi}\right)^2 + V(\boldsymbol{\phi}) \right\} \mathrm{d}^3x,$$
where the potential term is the standard baby Skyrme potential, $V(\boldsymbol{\phi}) = 2m^2(1 - \phi_3)$.
- Finite energy configurations require
$$\lim_{|x|\to\infty} \boldsymbol{\phi}(x) \equiv \boldsymbol{\phi}_\infty = \text{constant}.$$
- We must select $\boldsymbol{\phi}_\infty$ from the vacuum manifold of the model.
- For our choice of potential, we set $\boldsymbol{\phi}_\infty = (0, 0, 1)$.

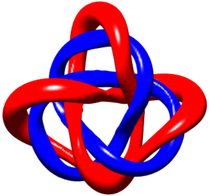
- This gives us the one-point compactification of space $\mathbb{R}^3 \cup \{\infty\} \cong S^3$.
- Each field can be characterized by the equivalence classes of the homotopy group $\pi_3(S^2) = \mathbb{Z}$.
- The topological charge $Q \in \pi_3(S^2) = \mathbb{Z}$ is the Hopf charge and finite energy field configurations are referred to as Hopfions.
- Let $F = \boldsymbol{\phi}^* \omega$ be the pullback of the area 2-form on the target 2-sphere to S^3.
- Triviality of the second cohomology group of the 3-sphere implies that F is exact, say $F = \mathrm{d}A$.
- The Hopf charge Q can be expressed as the integral of the Chern–Simons 3-form over the 3-sphere:

$$Q = \frac{1}{4\pi^2} \int_{S^2} F \wedge A.$$

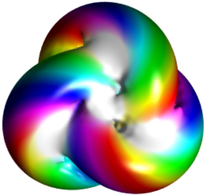
- There exists a lower bound on the static energy in terms of the Hopf charge Q.
- This cannot be attained by a Bogomolny-type argument, but is based on Sobolev-type inequalities. This was shown by Vakulenko & Kapitanski to be

$$E \geq cQ^{3/4}, \quad \text{where } c = \left(\frac{3}{16}\right)^{3/8}.$$

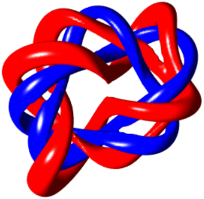
- As before, we create suitable initial field configurations for hopfions using rational maps $W : S^3 \to \mathbb{CP}^1$.
- Some example solutions are shown below. These show the linking structure between two independent points $(-1, 0, 0)$ and $(0, 0, -1)$ on the target 2-sphere, and the associated energy density isosurface.



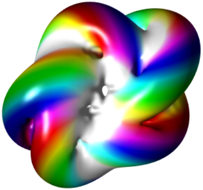
(a) $Q = 8 : \mathcal{K}_{3,2}$ knot linking



(b) $Q = 8 : \mathcal{K}_{3,2}$ knot energy density



(c) $Q = 12 : \mathcal{K}_{5,2}$ knot linking



(d) $Q = 12 : \mathcal{K}_{5,2}$ knot energy density

Table 2: Position (blue tube) and linking (red tube) curve for $m = 1$ Hopfions. These are preimages of the two cylinders defined by $\phi_3 = -1 + \epsilon$ and $\phi_1 = -1 + \epsilon$, where $\epsilon = 0.2$.