Nuclear matter as a crystal of topological solitons

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Solitons in the presence of vector mesons

- Large N_c-limit, QCD reduces to a weakly interacting theory of mesons, not only scalar but vector as well
- Can identify vector mesons with gauge multiplets of a minimally broken $\mathrm{SU}(2)_L \otimes \mathrm{SU}(2)_R \otimes \mathrm{U}(1)_V$ gauge model [*Phys. Rev. Lett.* **56**, 1035 (1986)]
- The ω-meson can be introduced by gauging the $\mathrm{U}(1)$ vector symmetry, and it couples anomalously through the gauged Wess-Zumino term
- Skyrme stabilizing term related to the effects of the ρ-meson field in the $m_\rho \to \infty$ limit
- Sextic term represents $m_\omega \to \infty$ limit of the ω-meson [*Phys. Lett. B* **145**, 101–106 (1985)]
- Natural to consider replacement of *adhoc* Skyrme term by explicit interactions with finite mass vector mesons

Solitons stabilized by ω-mesons

- Non-linear σ-model (NLσM) coupled to an ω-meson [*Phys. Lett. B* **137**, 251–256 (1984)]

$$\mathcal{L} = -\frac{1}{8\hbar^3}F_\pi^2 m_\pi^2 \mathrm{Tr}\left(\mathrm{Id}_2 - \varphi\right) - \frac{F_\pi^2}{16\hbar}\eta^{\mu\nu}\mathrm{Tr}(L_\mu L_\nu) + \frac{m_\omega^2}{2\hbar^3}\eta^{\mu\nu}\omega_\mu\omega_\nu - \frac{1}{4\hbar}\eta^{\mu\alpha}\eta^{\nu\beta}\omega_{\mu\nu}\omega_{\alpha\beta} + \beta_\omega\omega_\mu\mathcal{B}^\mu$$

- Wess-Zumino interaction term $\beta_\omega\omega_\mu\mathcal{B}^\mu$ describes coupling of the ω-meson to three pions
- Coupling constant β_ω related to the $\omega \to \pi^+\pi^-\pi^0$ decay rate
- Lagrangian is **singular** and static Lagrangian not bounded below $\Rightarrow$ non-trivial extremization
- In every Yang–Mills theory, the canonical momentum conjugate to the temporal component of the gauge field vanishes identically [*Nucl. Phys. A* **526**, 453–478 (1991)]: $p_0 = \frac{\partial \mathcal{L}}{\partial(\partial_0\omega_0)} = 0$
- Constitutes a primary constraint of the theory
- **Dirac–Bergmann algorithm** (singular Lagrangian $\to$ constrained Hamiltonian system): conservation of this primary constraint in time results in a secondary constraint of the form

$$\beta_\omega\mathcal{B}_0 + m_\omega^2\omega_0 - \partial_\mu p_\mu = 0 \quad \xrightarrow{\text{statics}} \quad \left(-\nabla^2 + m_\omega^2\right)\omega_0 = -\beta_\omega\mathcal{B}_0$$

- Can be solved and used to eliminate the constrained degree of freedom, ω_0

Geometric formulation of the ω-NLσM model

- Pion field: $\varphi : (M, g) \to (G, h)$
- M an oriented Riemannian manifold, and G a compact Riemannian manifold (normally $\mathrm{SU}(2)$)
- Omega meson $\omega = \omega_\mu \mathrm{d}x^\mu \in \Omega^1(M)$, $\omega_0 \in C^\infty(M)$
- Volume form $\Xi = \mathrm{vol}_h/|G| \quad \Rightarrow \quad \mathcal{B}_0 = *_g\varphi^*\Xi$
- Statics: constrained variational problem [*J. High Energ. Phys.* **07**, 184 (2020)]

$$E(\varphi, g) = \int_M \left(\frac{1}{8}|\mathrm{d}\varphi|^2 + \frac{1}{4}(V\circ\varphi) + \frac{1}{2}|\mathrm{d}\omega_0|^2 + \frac{1}{2}\omega_0^2\right)\mathrm{vol}_g \geq 0,$$

subject to the constraint

$$\left(\Delta_g + 1\right)\omega_0 = -c_\omega * \varphi^*\Xi, \quad c_\omega = \frac{m_\omega\beta_\omega}{F_\pi}$$

- Critical points of this are solutions of the Euler–Lagrange equations associated to the unconstrained Lagrangian $\mathcal{L}$
- The constraint can be solved by a non-linear conjugate gradient method

Varying the metric

- Consider the base space to be the 3-torus

$$(\mathbb{R}^3/\Lambda, g_{\mathrm{Euc}}), \quad \Lambda = \left\{n_1\vec{X}_1 + n_2\vec{X}_2 + n_3\vec{X}_3 : n_i \in \mathbb{Z}\right\}$$

- Key idea [*Comm. Math. Phys.* **332**, 355-377 (2014)]: Identify all 3-tori via diffeomorphism

$$F : (\mathbb{T}^3, g) \to (\mathbb{R}^3/\Lambda, g_{\mathrm{Euc}}), \quad x_1\vec{X}_1 + x_2\vec{X}_2 + x_3\vec{X}_3$$

- The metric on $\mathbb{T}^3 \equiv \mathbb{R}^3/\mathbb{Z}^3$ is the pullback $g = F^*g_{\mathrm{Euc}} = g_{ij}\mathrm{d}x^i\mathrm{d}x^j$, $g_{ij} = \vec{X}_i \cdot \vec{X}_j$
- $(\mathbb{T}^3, g)$ is equivalent to $(\mathbb{R}^3/\Lambda, g_{\mathrm{Euc}})$
- Vary metric g_s with $g_0 = F^*g_{\mathrm{Euc}} \iff$ vary lattice Λ_s with $\Lambda_0 = \Lambda$
- Energy minimized over variations of $g \iff$ optimal $\Lambda_\diamond$ [*Phys. Rev. D* **105**, 025010 (2022)]

ω-NLσM stress tensor

- To determine **crystalline phases**, we need to compute the **stress tensor**
- Given a smooth one-parameter family of variations (φ_s, g_s), soliton crystals are critical points of the energy $E(\varphi, g)$, that is solutions of [*Phys. Lett. B* **855**, 138842 (2024)]

$$\frac{\mathrm{d}}{\mathrm{d}s}E(\varphi_s, g_s)\bigg|_{s=0} = \int_{\mathbb{T}^3}\mathrm{d}^3x\sqrt{g}\,\left(\Phi_A(\varphi, g)\dot{\varphi}_A + S_{ij}(\varphi, g)\dot{g}_{kl}g^{jk}g^{li}\right) = 0$$

- $S(\varphi, g) \in \Gamma(\odot^2 T^*\mathbb{T}^3)$ is the stress tensor and $\Phi \in \Gamma(\varphi^{-1}T\mathrm{SU}(2))$ the tension field of φ
- The stress-energy tensor $S = S_{ij}\mathrm{d}x^i\mathrm{d}x^j$ associated to the energy

$$E(\varphi, g) = \int_M \left(\frac{1}{8}|\mathrm{d}\varphi|^2 + \frac{1}{4}(V\circ\varphi) + \frac{1}{2}|\mathrm{d}\omega_0|^2 + \frac{1}{2}\omega_0^2\right)\mathrm{vol}_g,$$

subject to the constraint

$$\left(\Delta_g + 1\right)\omega_0 = -c_\omega * \varphi^*\Xi,$$

is the section of $\mathrm{Sym}^2(T^*M)$ given by [*J. High Energ. Phys.* **06**, 116 (2024)]]

$$S(\varphi, g) = \left(\frac{1}{16}|\mathrm{d}\varphi|^2 + \frac{1}{8}(V\circ\varphi) - \frac{1}{4}|\mathrm{d}\omega_0|^2 - \frac{1}{4}\omega_0^2\right)g - \left(\frac{1}{8}\varphi^*h - \frac{1}{2}\mathrm{d}\omega_0\otimes\mathrm{d}\omega_0\right)$$

- Coincides with stress tensor of the unconstrained problem

Massive soliton crystals

- For fixed Skyrme $\mathcal{L}_{024}$-field φ, there **exists a unique critical point** of $E(\varphi, g)$ w.r.t. variations of g (generalizes to $\mathcal{L}_{0246}$-model) [*J. Math. Phys.* **64**, 103503 (2023)]
- Four crystals were found with $B_{\mathrm{cell}} = 4$: the $\varphi_{1/2}$, φ_α, φ_{chain} and $\varphi_{\mathrm{multiwall}}$ crystals
- From $\varphi_{1/2}$, the other three crystals can be constructed by applying a chiral $\mathrm{SO}(4)$ transformation $Q \in \mathrm{SO}(4)$, such that $\varphi = Q\varphi_{1/2}$, and minimizing $E(\varphi, g)$ w.r.t. variations of φ and g
- These are

$$Q \in \left\{\mathrm{Id}_4, \underbrace{\begin{pmatrix}(0,1,1,1)/\sqrt{3} \\ *\end{pmatrix}}_{Q_\alpha}, \underbrace{\begin{pmatrix}(0,0,0,1) \\ *\end{pmatrix}}_{Q_{\mathrm{multiwall}}}, \underbrace{\begin{pmatrix}(0,0,1,1)/\sqrt{2} \\ *\end{pmatrix}}_{Q_{\mathrm{chain}}}\right\}$$

- These four crystals were also found in the ω-NLσM model, with the ground state configuration dependent on the free parameters of the theory
- Related to symmetric scattering states of the $B = 4$ α-particle [*Phys. Lett. B* **391**, 150–156 (1997)]

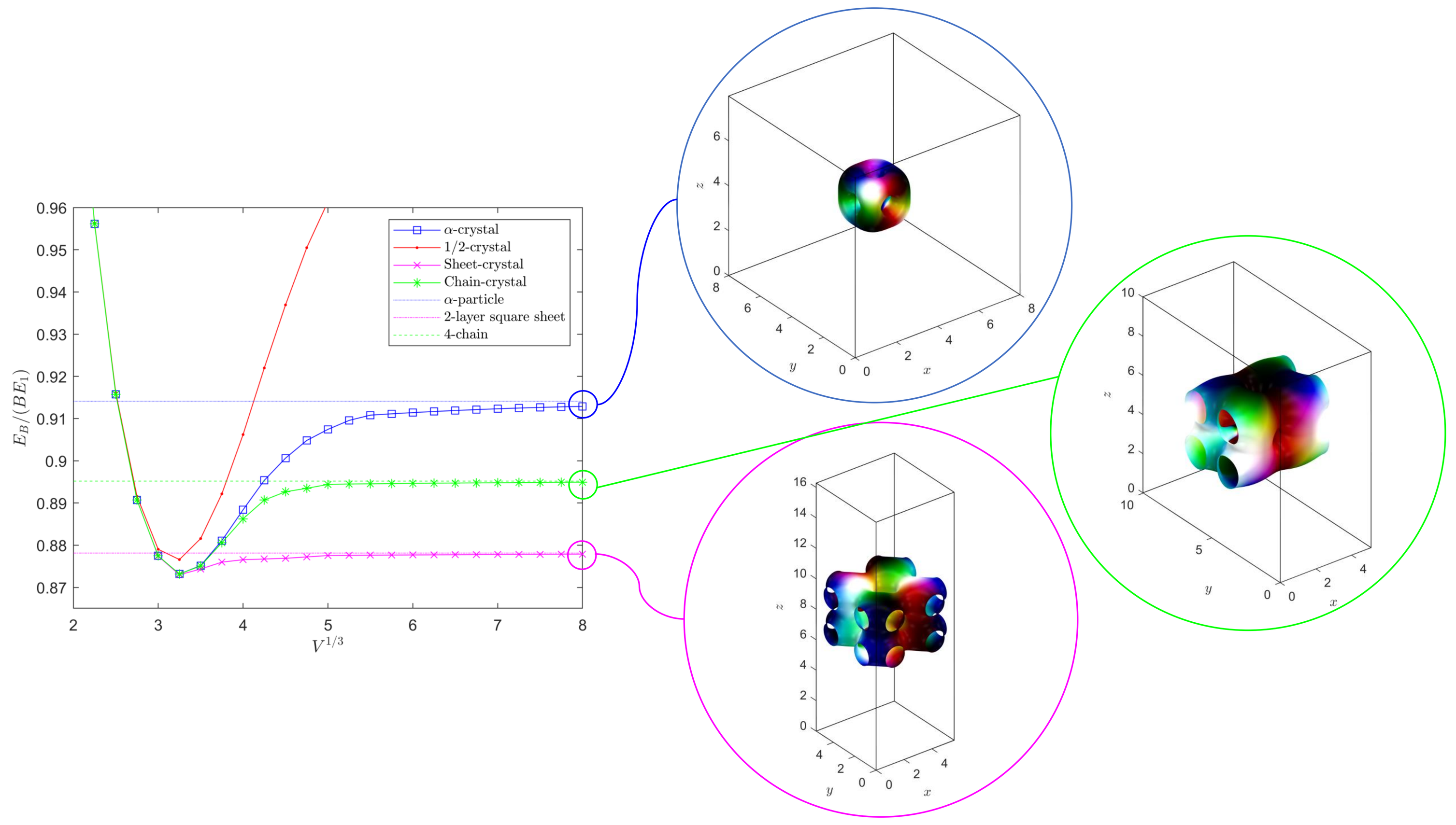

Figure 1. The E/B per unit cell of soliton crystals as a function of cell volume in the massive Skyrme model

Bethe–Weizsäcker semi-empirical mass formula

- Can use soliton crystals to estimate coefficients in the Bethe–Weizsäcker SEMF

$$E_b = a_V B - a_S B^{2/3} - a_C\frac{Z(Z-1)}{B^{1/3}} - a_A\frac{(N-Z)^2}{B} + \delta(N,Z), \quad \begin{cases} a_V \simeq 15.7 - 16.0\,\mathrm{MeV}, \\ a_S \simeq 17.3 - 18.4\,\mathrm{MeV}\end{cases}$$

- Method: approach the SEMF using α-particle approximation with n^3 α-particles
- Energy of a $B = 4n^3$ chunk in the α-particle approximation:

$$E(B) = \frac{E^\alpha_{\mathrm{crystal}}}{4}B + E_S^{\mathrm{chunk}}, \quad E_S^{\mathrm{chunk}} = 6n^2E^\alpha_{\mathrm{face}} = \frac{6E^\alpha_{\mathrm{face}}}{4^{2/3}}B^{2/3}$$

- Classical binding energy of an isospin symmetric chunk:

$$E_b = BE_1 - E(B) = \left(E_1 - \frac{E^\alpha_{\mathrm{crystal}}}{4}\right)B - \frac{3E^\alpha_{\mathrm{face}}}{\sqrt[3]{2}}B^{2/3} \quad \Rightarrow \quad \begin{cases} a_V = E_1 - \frac{1}{4}E^\alpha_{\mathrm{crystal}}, \\ a_S = \frac{3}{\sqrt[3]{2}}E^\alpha_{\mathrm{face}}\end{cases}$$

- Only need to compute the nucleon mass E_1, crystal energy $E^\alpha_{\mathrm{crystal}}$ and the energy of a single face of an α-particle E^α_{face}
- Previous results for $\mathrm{SC}_{1/2}$ crystal in the massless $\mathcal{L}_{24}$-Skyrme model: $a_V = 136\,\mathrm{MeV}$, $a_S = 320\,\mathrm{MeV}$ [*Nucl. Phys. A* **596**, 611–630 (1996)]
- Our calculations for the ω-NLσM model: $\boxed{a_V = 15.6\,\mathrm{MeV},\ a_S = 18.6\,\mathrm{MeV}}$

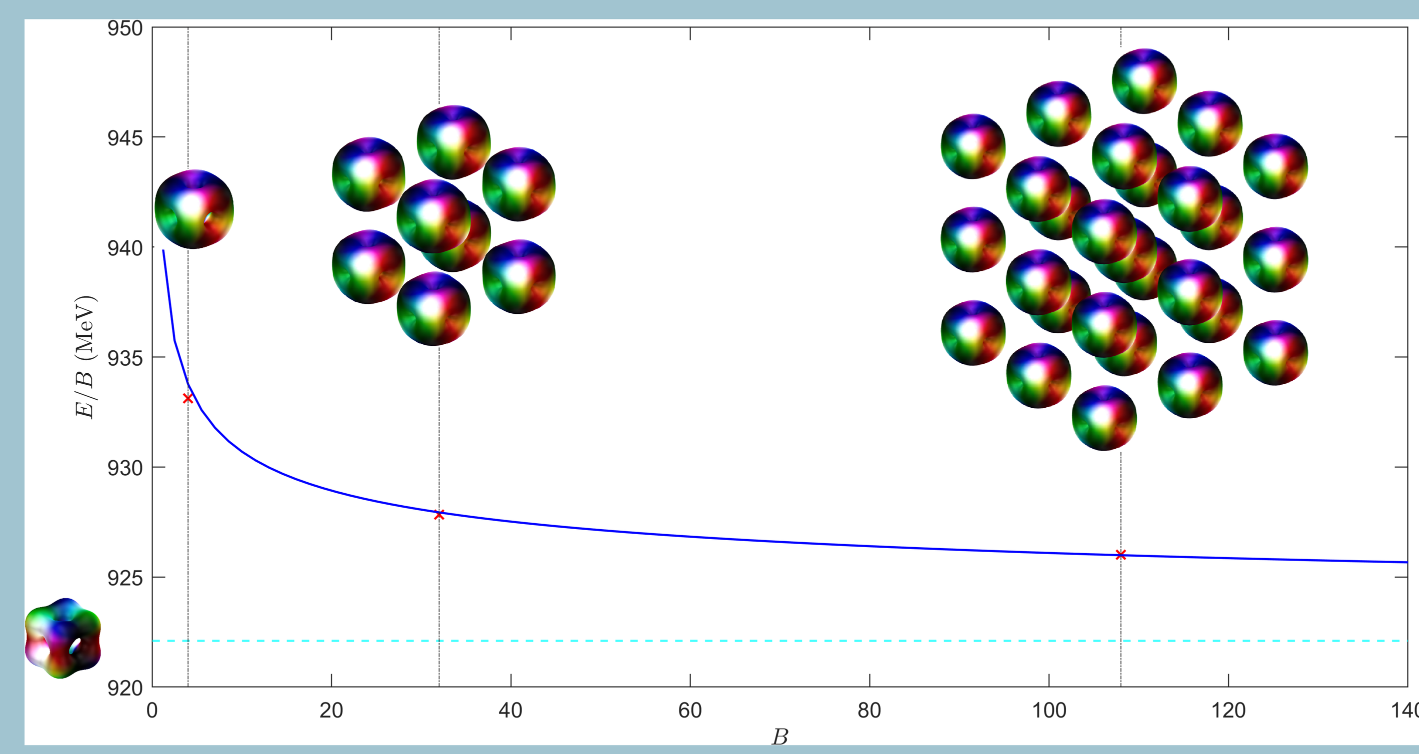

Figure 2. Plot of the Bethe–Weizsäcker SEMF from the α-particle approximation for the ω-NLσM model.

Compressibility of solitonic matter

- Energy of isospin symmetric nuclear matter [*Phys. Rev. D* **109**, 056013 (2024)]

$$E(n_B)/B = E_0 + \frac{1}{2}K_0\frac{(n_B - n_0)^2}{9n_0^2} + \mathcal{O}\left((n_B - n_0)^3\right), \quad K_0/E_0 \approx 0.260$$

- Approximate dense nuclear matter as a large extended crystalline configuration composed of N unit cells
- Each unit cell has baryon number B_{cell} and volume V_{cell}
- Baryon density is simply $n_B = NB_{\mathrm{cell}}/(NV_{\mathrm{cell}}) = B_{\mathrm{cell}}/V_{\mathrm{cell}}$
- In the thermodynamic limit $N \to \infty$, $E(n_B)/B \to E_{\mathrm{cell}}/B_{\mathrm{cell}}$
- E_{cell} is just the classical **static mass** of a soliton crystal $\to E_0 = E_{\mathrm{cell}}(n_0)/B_{\mathrm{cell}}$
- Can consider $E_{\mathrm{cell}}(n_B)$ by varying the baryon density n_B or, equivalently, the unit cell volume V_{cell}
- The compression modulus is thus

$$K_0 = \frac{9n_0^2}{B_{\mathrm{cell}}}\frac{\partial^2 E_{\mathrm{cell}}}{\partial n_B^2}\bigg|_{n_B=n_0} = \frac{9V_0^2}{B_{\mathrm{cell}}}\frac{\partial^2 E_{\mathrm{cell}}}{\partial V_{\mathrm{cell}}^2}\bigg|_{V_{\mathrm{cell}}=V_0}$$

- Fixed density variations requires replacing S_{ij} by its projection $S_{ij} - \frac{1}{\mathrm{Tr}_g(g)}S_{kl}g^{kl}g_{ij}$
- Previous results for $\mathrm{SC}_{1/2}$ crystal $K_0/E_0 \geq 1$ [*Phys. Rev. D* **90**, 045003 (2014)]
- Ground state crystal with the lowest B.E. coupling constant: multiwall with $c_\omega = 14.34$
- We find $E_0 = 917\,\mathrm{MeV}$ and $K_0 = 370\,\mathrm{MeV} \quad \Rightarrow \quad \boxed{K_0/E_0 = 0.403}$