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School of Mathematics and Statistics

Level-5 Honours project

Abelian Higgs Vortices on $\mathbb{H}^2$

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Abstract

The interactions of critically coupled vortices in the Abelian Higgs model are investigated for various geometries. We determine the form of the Kähler metric on the moduli space of static solutions for $\mathbb{R}^2$ and for compact Riemann surfaces M. It is demonstrated that the Bogomolny equations, for Abelian Higgs vortices, are integrable on the hyperbolic plane $\mathbb{H}^2$ of Gaussian curvature $-\frac{1}{2}$, this is achieved by a reduction of the Bogomolny equations to Liouville's equation. $\mathcal{N}$-vortex solutions can be geometrically constructed on $\mathbb{H}^2$ by considering holomorphic maps $f : \mathbb{H}^2 \to \mathbb{H}^2$, where the target space also has Gaussian curvature $-\frac{1}{2}$. One finds that the Higgs field ϕ is dependent on the the map f, and the ratio of the metrics on the domain and the target space. We observe that the map f is required to be a finite Blaschke rational function, where the ramification points of f are the positions of the vortex centres.

Contents

1 Introduction

There are essentially two types of solitons, integrable and topological (non-integrable), and their physical interpretations, as well as their fascinating mathematical properties, have been studied extensively in the literature. The basis of this paper is the study of topological solitons in Lorentz-invariant field theories. Integrable solitons take the form of localized lumps, whereas topological solitons take the form of domain walls, vortices, instantons, and monopoles. A static field theory consists of a set of field configurations, which are typically described by sections of fibre bundles or functions on some manifold, and an associated energy functional dependent on these fields. Field configurations which minimize the energy functional are solutions of the static field equations and, in particular, a topological soliton is a field configuration which minimizes the energy functional, and is topologically non-trivial. More exotic examples of topological solitons in field theories include sphalerons, which are saddle point solutions of the field equations, and calarons, which are chains of charge-2 instantons with axially-symmetry in Yang–Mills gauge theory. We will give a brief overview of topological solitons before discussing in detail vortices in the Abelian Higgs model.

1.1 Topological solitons

Although the static solutions of various non-integrable $(2+1)$-dimensional field theories have been explicitly constructed [17], a lot less is known about the dynamics of the solutions of such theories. However, there exists an analytic approach to studying the dynamics of solitons in non-integrable field theories – the geodesic approximation. The geodesic approximation (also referred to as the slow-motion, adiabatic, or moduli space approximation) was originally proposed by Manton [10] in his paper on the scattering of BPS monopoles. This is a powerful method for investigating the slow-motion scattering of monopoles, in which the dynamics can be described by considering the dynamics in a reduced, finite-dimensional parameter space of static solutions, the *moduli space* [16]. The moduli space is parameterized by collective coordinates, with its metric being induced by the kinetic energy functional of the gauged field theory. This moduli space approximation serves as a good approximation provided the velocities are sufficiently low such that the slowly-moving monopoles are restricted to motion tangent to the moduli space. The resulting dynamical problem is reduced to calculating the geodesic motion on the moduli space with respect to the induced metric. Thus it is essential to obtain the static solutions of the underlying theory, in order to determine the metric on the moduli space.

One particular model known to admit static solutions is the Abelian Higgs model at critical coupling. The corresponding solutions are known as vortices and forces between these vortices balance at the critical coupling value, giving rise to mulit-vortex solutions. The static Abelian Higgs theory can be used to describe the Ginzburg–Landau theory of superconductivity, where the critical coupling value distinguishes between Type I and Type II superconductivity. The Abelian Higgs model can also be used to describe the interaction of $U(1)$ gauged cosmic strings in cosmology. This model exhibits the Higgs mechanism and the spontaneous-symmetry breaking results in the formation of these vortex strings during the cooling of the early universe [12].

In a remarkable paper by Witten [1], Witten constructed all the self-dual $SU(2)$ Yang-Mills instantons on $\mathbb{R}^4$, which are invariant under an $SO(3)$ symmetry. In particular Witten showed that, via the $SO(3)$ symmetry, the self-duality equations in $\mathbb{R}^4$ reduce to the Bogomolny equations for Abelian Higgs vortices on the hyperbolic plane $\mathbb{H}^2$ with Gaussian curvature $-\frac{1}{2}$. This construction gives a Yang-Mills gauge field on the product manifold $\mathbb{H}^2 \times S^2$, which can be extended smoothly to a gauge field on $\mathbb{R}^4$ [18]. Furthermore, Witten showed that the Abelian Higgs vortices on the Hyperbolic plane $\mathbb{H}^2$ are in fact integrable, as the Bogomolny equations in this case reduce to the integrable Liouville's equation. The explicit solutions of this can be constructed by considering a holomorphic mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$. One finds that f must be a finite Blaschke rational product, so that the solutions satisfy the boundary conditions and give a finite-valued Yang–Mills action.

This paper is laid out as follows. We begin by laying the foundations of classical field theory before discussing gauge field theories. Emphasis is placed on $U(1)$ gauge field theories and the Lagrangian for scalar electrodynamics is derived, both for flat spacetimes and for curved spacetimes. Spontaneous-symmetry breaking and the Higgs mechanism are then discussed, thereby naturally introducing the Abelian Higgs model. A brief discussion on Derrick's theorem and the existence of non-vacuum solutions then follows. This introduces the formalism required to investigate vortex solutions of the Abelian Higgs model for various geometries. In Section 3, Abelian Higgs vortices on $\mathbb{R}^2$ and its relation to the Ginzburg–Landau theory of superconductivity is presented. Then the dynamics of critically coupled vortices are discussed, naturally leading to the introduction of the geodesic approximation. The subsequent dynamics on the moduli space and the form of the metric on the moduli space are detailed. Section 4 introduces a variation on the geometry and, in particular, Abelian Higgs vortices on the hyperbolic plane $\mathbb{H}^2$ are constructed. Finally, a brief conclusion and a discussion on possible future work are presented.

1.2 Notation and convention

Throughout this paper we will adopt the following convention and notation. Roman indices run over space $i, j, k = 1, 2, \ldots, d$, whereas Greek indices run over spacetime $\mu, \nu = 0, 1, 2, \ldots, d$. A covariant vector is of the form $x^\mu = (x^0, x^1, \ldots, x^d)$, where one normally drops the indice and it is assumed that $x = x^\mu$, unless stated otherwise. We will also sometimes write this as $x = (x^0, \mathbf{x})$ or $x = (t, \mathbf{x})$. Similarly, a contravariant vector takes the form $x_\mu = (x^0, -x^1, \ldots, -x^d)$. We adopt the Einstein summation convention, i.e. summing over repeated indices. For example,

$$x_\mu y^\mu = \sum_{\mu=0}^{d} x_\mu y^\mu, \quad F_{\mu\nu}F^{\mu\nu} = \sum_{\mu,\nu=0}^{d} F_{\mu\nu}F^{\mu\nu}. \tag{1.1}$$

The signature of a $(d+1)$-dimensional Minkowski spacetime is always $(+, -, \cdots, -)$. On a flat $(d+1)$-dimensional Minkowski spacetime $\mathbb{R}\times\mathbb{R}^d$, the Minkowski metric tensor is $g_{\mu\nu} = \mathrm{diag}(+, -, \cdots, -)$ and the metric is given by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = (dx^0)^2 - (dx^i)^2. \tag{1.2}$$

The flat Minkowski spacetime $\mathbb{R} \times \mathbb{R}^d$ is equipped with the inner product

$$xy = x^\mu y_\mu = x_\mu y^\mu = g_{\mu\nu}x^\mu y^\nu = x^0 y^0 - x^i y^i. \tag{1.3}$$

The following notations for derivatives are employed throughout

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\partial_0, \nabla), \quad \partial^\mu = \frac{\partial}{\partial x_\mu} = (\partial_0, -\nabla). \tag{1.4}$$

If we now consider a $(2 + 1)$-dimensional spacetime and introduce the local complex coordinate $z = x^1 + ix^2$, then it is understood that the complex derivatives are

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{\partial}{\partial \bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2), \tag{1.5}$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

It is also worth noting that a Lie group is written as G and its associated Lie algebra is denoted by the lowercase math fraktur font $\mathfrak{g}$. For example, the $U(1)$ Lie group has the Lie algebra $\mathfrak{u}(1)$ and similarly $SO(2)$ has the Lie algebra $\mathfrak{so}(2)$.

2 Field theory

The purpose of this section is to concisely introduce the required field theory formalism necessary for the discussion of vortex solutions in the Abelian Higgs model. In particular, our aim is to discuss $U(1)$ gauge field theory and present the scalar electrodynamics Lagrangian for flat and curved spacetimes. From here we can introduce the symmetry-breaking Higgs potential and discuss the rôle of the Higgs mechanism in the theory.

2.1 Classical field theory

In classical Lagrangian field theory, one is concerned with the dynamics of fields throughout spacetime. We will only consider scalar fields ϕ in this paper, locally denoted as $\phi(x)$. Essentially, a field ϕ can be considered as representing an infinite number of dynamical degrees of freedom, where the value of the field at a spatial point represents one degree of freedom, which evolves in time [2].

We can describe the dynamics of classical fields using a variational formalism, and by considering the Lagrangian and the action principle. The evolution of a system progresses along the path of stationary action, where the action S is defined in terms of the Lagrangian L.

2.1.1 Lagrangian field theory

Let us first suppose that space is $\mathbb{R}^d$ so that spacetime is $\mathbb{R} \times \mathbb{R}^d$. Then the action S associated with the Lagrangian density $\mathcal{L}(\phi, \partial_\mu \phi)$ is given by

$$S[\phi] = \int L\,dt = \int \int_{\mathbb{R}^d} \mathcal{L}(\phi, \partial_\mu \phi)\, d^d x\, dt, \tag{2.1}$$

where the integration is over spacetime.

The action principle states that a system evolves between initial data $\phi(t_1, \mathbf{x})$ and final data $\phi(t_2, \mathbf{x})$ along the "path" in configuration space which extremizes (normally minimizes) the action S. In other words, the field will adopt the configuration which reduces the action S to its minimum. To find this configuration, we look for a field configuration such that an infinitesimally small variation of the field leaves the action unchanged, i.e.

$$\phi(x) \mapsto \phi'(x) = \phi(x) + \delta\phi(x) \quad \Longrightarrow \quad S \mapsto S + \delta S \quad \text{with} \quad \delta S = 0, \tag{2.2}$$

where

$$\delta S = \int \int_{\mathbb{R}^d} \left(\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu(\delta\phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) d^d x\, dt. \tag{2.3}$$

The variations satisfy the following boundary conditions, $\delta\phi \to 0$ as $|\mathbf{x}| \to \infty$ and $\delta\phi(t_1, \mathbf{x}) = \delta\phi(t_2, \mathbf{x}) = 0$. Now, using the following relation

$$\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = \partial_\mu(\delta\phi) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} + \delta\phi\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \tag{2.4}$$

and integrating by parts yields

$$\delta S = \int \int_{\mathbb{R}^d} \left[\delta\phi \frac{\partial \mathcal{L}}{\partial \phi} - \delta\phi\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] d^d x\, dt + \int \int_{\mathbb{R}^d} \left[\partial_\mu \left(\delta\phi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) \right] d^d x\, dt. \tag{2.5}$$

The last term can be turned into a surface integral over the boundary of the $(d + 1)$-dimensional spacetime region of integration [8]. By virtue of the boundary conditions, this surface term vanishes.

This has to be true (for ϕ to be an extremum of S) for all $\delta\phi$, so for the variation in the action δS to vanish we require

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) = 0. \tag{2.6}$$

This is the Euler–Lagrange equations of motion for a dynamical field, and if there are multiple fields then there is one such equation for each field.

The simplest Lorentz invariant field theory in a $(d+1)$-dimensional Minkowski spacetime $\mathbb{R} \times \mathbb{R}^d$ has the Lagrangian

$$L[\phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - U(\phi) \right) d^d x, \tag{2.7}$$

where $U(\phi)$ is a potential (with no explicit $\mathbf{x}$ dependence), normally taken to be a polynomial. We can split this Lagrangian up into its corresponding kinetic energy term,

$$T[\phi] = \frac{1}{2} \int_{\mathbb{R}^d} (\partial_0 \phi)^2 \, d^d x, \tag{2.8}$$

and potential energy term,

$$V[\phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2} \nabla \phi \cdot \nabla \phi + U(\phi) \right) d^d x, \tag{2.9}$$

such that $L = T - V$. The Euler-Lagrange equations corresponding to the Lagrangian (2.7) read

$$\partial_0 \partial_0 \phi - \nabla^2 \phi + \frac{dU}{d\phi} = \Box \phi + \frac{dU}{d\phi} = 0, \tag{2.10}$$

where $\Box = \partial_\mu \partial^\mu = \partial_0 \partial_0 - \nabla^2$ is the Laplace operator of Minkowski spacetime [9].

We can also extend this theory to a spacetime $\mathbb{R} \times M$ where the space M is endowed with a Riemannian metric $h_{ij}(\mathbf{x})$. The integration measure is $\sqrt{\det h} \, d^d x$, which is the natural measure on M, and the term $\nabla \phi \cdot \nabla \phi$ simply becomes $h^{ij} \partial_i \phi \partial_j \phi$. The Lagrangian (2.7) then becomes

$$L[\phi] = \int_M \left(\frac{1}{2} (\partial_0 \phi)^2 - \frac{1}{2} h^{ij} \partial_i \phi \partial_j \phi - U(\phi) \right) \sqrt{\det h} \, d^d x \tag{2.11}$$

and the Euler-Lagrange field equations now read

$$\partial_0 \partial_0 \phi - \frac{1}{\sqrt{\det h}} \partial_i \left(\sqrt{\det h} \, h^{ij} \partial_j \phi \right) + \frac{dU}{d\phi} = 0. \tag{2.12}$$

2.1.2 Noether's Theorem

Noether's theorem says that if the action S is unchanged under a transformation, then there exists a conserved current $j^\mu(x)$ associated with the symmetry such that the equation of motion implies that

$$\partial_\mu J^\mu = 0. \tag{2.13}$$

More explicitly, this gives

$$\partial_0 \rho + \nabla \cdot \mathbf{j} = 0, \tag{2.14}$$

where $\rho = J^0$ is a conserved charge density and $\mathbf{j} = (J^i)$ is a current density [5].

Consider an arbitrary infinitesimal transformation of the field

$$\phi(x) \mapsto \phi(x) + \delta\phi(x), \tag{2.15}$$

where $\delta\phi(x)$ is some infinitesimal deformation of the field configuration [8]. This variation is known as a symmetry if it leaves the Euler–Lagrange equations, and thus the action, invariant. In fact, one can show that this variation is a symmetry if the corresponding variation of the Lagrangian density $\mathcal{L}$ is a total divergence [2],

$$\mathcal{L}(x) \mapsto \mathcal{L}(x) + \partial_\mu \mathcal{J}^\mu(x), \tag{2.16}$$

for some $\mathcal{J}^\mu(x)$. Let us explicitly compute the change in $\mathcal{L}$,

$$\begin{aligned} \delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) \\ &= \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right). \end{aligned} \tag{2.17}$$

The first term vanishes by the Euler-Lagrange equations and thus have

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right). \tag{2.18}$$

But by the definition of symmetry transformation, we set the remaining term equal to $\partial_\mu\mathcal{J}^\mu(x)$ and find

$$\partial_\mu J^\mu = 0 \tag{2.19}$$

for

$$J^\mu(x) = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \mathcal{J}^\mu \tag{2.20}$$

This result states that the current J^μ is conserved; moreover, it also tells us what it is. There is a conservation law for each continuous symmetry of $\mathcal{L}$.

2.1.3 The energy-momentum tensor

Recall that in classical particle mechanics, invariance under spatial translations gives rise to the conservation of momentum, while invariance under time translations is responsible for the conservation of energy. We will now see something similar in field theories.

Noether's theorem can also be applied to spacetime transformations such as translations and rotations. Consider the infinitesimal translation in Minkowski spacetime

$$x^\mu \mapsto x^\mu - \epsilon^\mu,$$

this induces an active transformation of the field configuration

$$\phi(x) \mapsto \phi(x+\epsilon) = \phi(x) + \epsilon^\mu\partial_\mu\phi(x).$$

Since the Lagrangian density is also a scalar, it transforms as

$$\mathcal{L} \to \mathcal{L} + \epsilon^\mu\partial_\mu\mathcal{L} = \mathcal{L} + \epsilon^\nu\partial_\mu(\delta^\mu{}_\nu\mathcal{L}).$$

Since the change in the Lagrangian is a total derivative, we may invoke Noether's theorem which gives us d conserved currents $(j^\mu)_\nu$, one for each of the translations ϵ^ν with $\nu = 0, 1, ..., d$,

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi - g^\mu{}_\nu\mathcal{L} \equiv T^\mu{}_\nu. \tag{2.21}$$

The conserved current is the stress-energy tensor, also called the it energy-momentum tensor, of the field ϕ. It satisfies

$$\partial_\mu T^\mu{}_\nu = 0. \tag{2.22}$$

We can also write the energy-momentum tensor as

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial^\nu \phi - g^{\mu\nu}\mathcal{L} \quad T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu}\mathcal{L}. \tag{2.23}$$

The conserved charge associated with time translations is the energy,

$$E[\phi] = \int_{\mathbb{R}^d} T^{00}\, d^d x = \int_{\mathbb{R}^d} \left[\frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial^0 \phi - \mathcal{L} \right] d^d x, \tag{2.24}$$

and the conserved charges associated with spatial translations are the momentum components

$$P_i[\phi] = -\int_{\mathbb{R}^d} T_i^0\, d^d x = -\int_{\mathbb{R}^d} \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} \partial_i \phi\, d^d x. \tag{2.25}$$

In analogy to point particle mechanics, we thus see that invariants of the Lagrangian density correspond to conservation laws. An entirely analogous procedure leads to conserved quantities like angular momentum. Furthermore, one can study so-called internal symmetries, i.e., ones which are not related to coordinate but other transformations.

2.2 Gauge field theory

A gauge theory is a type of field theory where an internal Lie symmetry group G acts locally. Gauge transformations form a Lie group and if field configurations only differ by a gauge transformation then they are regarded as physically the same. We refer to this Lie group as the gauge group, or symmetry group, of the theory.

We begin by introducing an abelian gauge theory with the symmetry group $G = U(1)$, which physically describes the interaction of fields with the electromagentic field. Throughout this section we shall consider a complex scalar field $\phi(x) = \phi_1(x) + i\phi_2(x)$ coupled to an electromagnetic field with symmetry group $U(1)$. But first, consider the ungauged theory in Minkowski spacetime $\mathbb{R} \times \mathbb{R}^d$ with the Lagrangian

$$L[\phi] = \int_{\mathbb{R}^d} \left(\frac{1}{2} \partial_\mu \bar{\phi} \partial^\mu \phi - U(|\phi|^2) \right) d^d x. \tag{2.26}$$

It is trivial to show that the Lagrangian (2.26) is invariant under the *global* phase rotation

$$\phi(x) \mapsto e^{iq\xi} \phi(x), \tag{2.27}$$

where we have introduced the parameter q, which measures the strength of the phase transformations [5]. Hence, there exists an internal symmetry $U(1) \cong SO(2)$.

Now, let us consider *local* phase transformations. In order to obtain a $U(1)$ gauge theory, we require the Lagrangian to be invariant under the gauge transformation

$$\phi(x) \mapsto e^{iq\xi(x)} \phi(x). \tag{2.28}$$

To construct such a Lagrangian, we need to extend the global gauge invariance to a local gauge invariance. (It is worth noting that the potential $U(|\phi|^2)$ is already gauge invariant.) Under the local phase transformation (2.28), the derivative $\partial_\mu \phi(x)$ will be subject to transformations at neighbouring spacetime points, which results in the following effect:

$$\begin{aligned} \partial_\mu \phi(x) &\mapsto \partial_\mu \left(e^{iq\xi(x)} \phi(x) \right) \\ &= e^{iq\xi(x)} \left(\partial_\mu \phi(x) + iq \partial_\mu \xi(x) \phi(x) \right). \end{aligned} \tag{2.29}$$

In order to make a Lagrangian invariant under local phase transformations, we want to introduce new terms whose variations will compensate for the additional $\partial_\mu \xi(x)$ term. We do this by introducing a *covariant derivative* D_μ which transforms covariantly according to

$$D_\mu \phi(x) \mapsto e^{iq\xi(x)} D_\mu \phi(x). \tag{2.30}$$

Comparing (2.29) and (2.30), we see that we can compensate for the $\partial_\mu \xi(x)$ term by defining the gauge covariant derivative

$$D_\mu \phi(x) = \left(\partial_\mu - iqA_\mu(x)\right)\phi(x), \tag{2.31}$$

where $A_\mu(x)$ is a covariant rank 1 tensor. Then, applying the gauge transformation (2.28) to the covariant derivative (2.31) we obtain

$$D_\mu \phi(x) \mapsto e^{iq\xi(x)} \left(\partial_\mu \phi(x) + iq\partial_\mu \xi(x)\phi(x) - iqA'_\mu \phi(x)\right). \tag{2.32}$$

For the covariant derivative (2.31) to be covariant, we require A_μ to obey the transformation rule

$$A_\mu(x) \mapsto A'_\mu(x) = A_\mu(x) + \partial_\mu \xi(x). \tag{2.33}$$

Hence, we have introduced a new field A_μ in order to construct a gauge invariant Lagrangian. This new field A_μ is known as the electromagnetic gauge potential, and is called a *gauge field*.

Since A_μ is real, we have that the covariant derivative of the complex scalar field $\bar{\phi}$ is simply the complex conjugate of the covariant derivative of ϕ, i.e.

$$\overline{D_\mu \phi} = \partial_\mu \bar{\phi} + iqA_\mu \bar{\phi}. \tag{2.34}$$

Thus, under the gauge transformations (2.28) and (2.33), $\overline{D_\mu \phi}$ transforms as

$$\overline{D_\mu \phi} \mapsto e^{-iq\xi(x)} \overline{D_\mu \phi}. \tag{2.35}$$

Therefore, the term $\overline{D_\mu \phi} D^\mu \phi$ is gauge invariant and and we can add it to our construction of a gauge invariant Lagrangian. Our Lagrangian density in (2.26) simply becomes

$$\mathcal{L}_\phi = \frac{1}{2}\overline{D_\mu \phi} D^\mu \phi - U(|\phi|^2). \tag{2.36}$$

Before we go onto discuss dynamical terms involving derivatives of our gauge field A_μ, we digress briefly to discuss some properties of the covariant derivative D_μ. This will naturally lead us to introduce the curvature tensor $F_{\mu\nu}$, and enable us to write a gauge invariant Lagrangian term for the gauge field. We can view the covariant derivative D_μ as consisting of two terms, each of which is related to an infinitesimal transformation. The first term ∂_μ generates an infinitesimal displacement $x^\mu \to x^\mu + \epsilon^\mu$, whereas the second term $-iqA_\mu$ generates a field-dependent gauge transformation with parameter $\xi = -\epsilon^\mu A_\mu$. Their combination is referred to as a covariant translation and, under this translation, a field transforms as [6]

$$\delta\phi = \epsilon^\mu D_\mu \phi. \tag{2.37}$$

This shows that the covariant derivative D_μ corresponds to an infinitesimal variation, which must satisfy the Liebniz rule:

$$D_\mu(\phi_1\phi_2) = (D_\mu\phi_1)\phi_2 + \phi_1(D_\mu\phi_2). \tag{2.38}$$

One can verify this by considering local phase transformations with strengths q_1 and q_2. Then we have

$$\begin{aligned} D_\mu(\phi_1\phi_2) &= (\partial_\mu - i(q_1+q_2)A_\mu)(\phi_1\phi_2) \\ &= (\partial_\mu\phi_1 - iq_1A_\mu\phi_1)\phi_2 + \phi_1(\partial_\mu\phi_2 - iq_2A_\mu\phi_2) \\ &= (D_\mu\phi_1)\phi_2 + \phi_1(D_\mu\phi_2), \end{aligned}$$

as required.

Now we want to consider the non-commutativity of the covariant derivative D_μ, by applying the antisymmetric product

$$[D_\mu, D_\nu]\phi = D_\mu(D_\nu\phi) - D_\nu(D_\mu\phi). \tag{2.39}$$

Writing out the first term explicitly we have

$$D_\mu(D_\nu\phi) = \partial_\mu\partial_\nu\phi - iqA_\mu\partial_\nu\phi - iq(\partial_\mu A_\nu)\phi - iqA_\nu\partial_\mu\phi - q^2A_\mu A_\nu\phi. \tag{2.40}$$

From this, we arrive at the commutation relation

$$[D_\mu, D_\nu] = -iqF_{\mu\nu}, \tag{2.41}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.42}$$

is the curvature 2-tensor [5]. This is a measure of the non-commutativity of the covariant derivaties. The commutation relation (2.41) is called the Ricci identity. Introducing the curvature tensor (2.42) ensures that the fields A_μ are dynamical. From the Ricci identity (2.41), we see that $F_{\mu\nu}$ must transform covariantly. Further, applying the gauge transformation (2.33), we see that the curvature tensor is also gauge invariant,

$$\begin{aligned} F_{\mu\nu} &\mapsto \partial_\mu(A_\nu + \partial_\nu\xi) - \partial_\nu(A_\mu + \partial_\mu\xi) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= F_{\mu\nu}. \end{aligned} \tag{2.43}$$

However, this is related to the fact that our gauge transformations are Abelian.

We also have that covariant derivatives satisfy the Jacobi identity,

$$[D_\mu, [D_\nu, D_\rho]] + [D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] = 0. \tag{2.44}$$

Expanding this identity out and explicitly writing the double commutator of covariant derivatives,

$$[D_\mu, [D_\nu, D_\rho]]\phi = -iq(D_\mu F_{\nu\rho})\phi, \tag{2.45}$$

leads to the following identity:

$$D_\mu F_{\nu\rho} + D_\rho F_{\mu\nu} + D_\nu F_{\rho\mu} = 0. \tag{2.46}$$

As the curvature tensor is gauge invariant, we can simply express this in terms of ordinary derivatives, which yields the Bianchi identity,

$$\partial_\mu F_{\nu\rho} + \partial_\rho F_{\mu\nu} + \partial_\nu F_{\rho\mu} = 0. \tag{2.47}$$

The Bianchi identity (2.47) implies that we can express the curvature tensor $F_{\mu\nu}$ in terms of a vector field [6], which is in accordance with (2.42). We can identify the components of $F_{\mu\nu}$ with the electric field $\mathbf{E} = (E_i)$ and the magentic field $\mathbf{B} = (B_i)$. The electric components of the field tensor $F_{\mu\nu}$ are given by

$$F_{0i} = E_i = \partial_0 A_i - \partial_i A_0, \tag{2.48}$$

which comprise a 1-form in space, and the magnetic components of the field tensor are given by

$$F_{ij} = -\varepsilon_{ijk}B_k = \partial_i A_j - \partial_j A_i, \tag{2.49}$$

which comprise a 2-form in space. Thus, the curvature tensor $F_{\mu\nu}$ is commonly referred to as the *electromagnetic field tensor*. We can also express the field tensor $F_{\mu\nu}$ in matrix form,

$$(F_{\mu\nu}) = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & -B_3 & B_2 \\ -E_2 & B_3 & 0 & -B_1 \\ -E_3 & -B_2 & B_1 & 0 \end{pmatrix}. \tag{2.50}$$

The electromagnetic field tensor $F_{\mu\nu}$ can be used to construct a gauge invariant Lagrangian term for the gauge field A, via the Lorentz scalar $F_{\mu\nu}F^{\mu\nu}$ [5], which is given by

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{2.51}$$

By combining (2.36) and (2.51), we are now able to construct a Lagrangian density $\mathcal{L} = \mathcal{L}_A + \mathcal{L}_\phi$ for scalar electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - U(|\phi|^2), \tag{2.52}$$

which is invariant under Lorentz transformations and the combined gauge transformations (2.28) and (2.33). The requirement of a local gauge invariance results in an interacting field theory, where scalar electrodynamics describes the theory of photons coupled to charged spinless fields. One can separate this Lagrangian density into space and time components, which yields

$$\mathcal{L} = \frac{1}{2}E_iE_i + \frac{1}{2}\overline{D_0\phi}D_0\phi - \frac{1}{4}F_{ij}F_{ij} - \frac{1}{2}\overline{D_i\phi}D_i\phi - U(|\phi|^2). \tag{2.53}$$

Now we can define the kinetic and potential energy functionals,

$$T[\phi, A] = \int_{\mathbb{R}^d} \left(\frac{1}{2}E_iE_i + \frac{1}{2}\overline{D_0\phi}D_0\phi\right) d^dx \tag{2.54}$$

and

$$V[\phi, A] = \int_{\mathbb{R}^d} \left(\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\overline{D_i\phi}D_i\phi + U(|\phi|^2)\right) d^dx, \tag{2.55}$$

respectively. The sign convention in the Lagrangian density (2.52) is chosen such that the kinetic energy (2.54) is positive definite. Note that the potential energy expression of scalar electrodynamics (2.55) is called the Ginzburg–Landau free energy, which we will explore in detail in Section 3 when we look at vortex solutions in the Abelian Higgs model. The reason for labeling the potential energy as the Ginzburg–Landau free energy will become transparent at the end of this section. From the energy-momentum tensor, the Hamiltonian of the scalar electrodynamics is given by

$$\begin{aligned} E[\phi, A] = \int_{\mathbb{R}^d} \mathcal{H}\, d^dx &= \int_{\mathbb{R}^d} \left(\frac{\partial\mathcal{L}}{\partial(\partial_0\phi)}\partial_0\phi + \frac{\partial\mathcal{L}}{\partial(\partial_0\bar{\phi})}\partial_0\bar{\phi} + \frac{\partial\mathcal{L}}{\partial(\partial_0 A_\sigma)}\partial_0 A_\sigma - \mathcal{L}\right) d^dx \\ &= \int_{\mathbb{R}^d} \left(\frac{1}{2}E_iE_i + \frac{1}{2}\overline{D_0\phi}D_0\phi\right) d^dx + \int_{\mathbb{R}^d} \left(\frac{1}{4}F_{ij}F_{ij} + \frac{1}{2}\overline{D_i\phi}D_i\phi + U(|\phi|^2)\right) d^dx \\ &= T + V. \end{aligned} \tag{2.56}$$

So the total energy of the system is simply the kinetic energy plus the potential energy. When we consider Derrick's theorem in Section 2.4, it will be more useful to have the Hamiltonian density in the following form:

$$\begin{aligned} \mathcal{H} &= \frac{1}{2}\left(|\mathbf{B}|^2 + |\mathbf{E}|^2\right) + \frac{1}{2}\left(|D_0\phi|^2 + |\mathbf{D}\phi|^2\right) + U(|\phi|^2) \\ &= \frac{1}{4}|F|^2 + \frac{1}{2}|D\phi|^2 + U(|\phi|^2). \end{aligned} \tag{2.57}$$

One can readily obtain the field equations for Lorentz invariant scalar electrodynamics, with Lagrangian density given by Eq. (2.52), by considering variations of fields $\{\phi, A_\mu\}$. We require the condition that the action functional,

$$S[\phi, A] = \int\int_{\mathbb{R}^d} \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - U(|\phi|^2)\right) d^dx\, dt,$$

must be invariant under these variations. Here we assume that the variations $\{\delta\phi, \delta A_\mu\}$ vanish at spatial infinity and at the temporal beginning and end. Then the field equations (obtained by taking variations with respect to the fields $\{\bar{\phi}, A_\nu\}$) are given by

$$\frac{\partial\mathcal{L}}{\partial\bar{\phi}} - D_\mu\left(\frac{\partial\mathcal{L}}{\partial(\overline{D_\mu\phi})}\right) = 0 \quad \Rightarrow \quad \Box_D\phi = -2U'(|\phi|^2)\phi, \tag{2.58}$$

$$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) = 0 \quad \Rightarrow \quad \partial_\mu F^{\mu\nu} = -J^\nu, \tag{2.59}$$

where $\Box_D = D_\mu D^\mu$ is the covariant d'Alembert operator and

$$J^\nu = \frac{iq}{2}\left(\bar{\phi}D^\nu\phi - \phi\overline{D^\nu\phi}\right) \tag{2.60}$$

is the Noether current associated with the $U(1)$ global symmetry (2.27). The conservation law $\partial_\nu J^\nu = 0$ is consistent with (2.59) and is a direct consequence of (2.58). Recall from Section 2.1.2 that $J^0 = \rho$ is the conserved charge density and $\mathbf{j} = (J^i)$ is the current density. Then, consider the $\nu = 0$ component of the field equation (2.59), which comes from variation of the A_0 field. This yields Gauss' law $\nabla \cdot \mathbf{E} = \rho$, which implies that we should identify

$$\rho = J^0 = \frac{iq}{2}\left(\bar{\phi}D_0\phi - \phi\overline{D_0\phi}\right) \tag{2.61}$$

with the electric charge density. It is worth noting that there is no electric charge unless our field ϕ is complex-valued; real-valued fields ϕ describe electrically neutral particles [5].

We can expand out Gauss' law (2.61) and, using the field equation (2.59), express it as

$$(\nabla^2 - q|\phi|^2)A_0 = \partial_i\partial_0 A_i + \frac{iq}{2}(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}). \tag{2.62}$$

It is sometimes possible to fix the gauge such that the right-hand side above vanishes, and we can then choose $A_0 = 0$ [2].

Now, consider the static situation of scalar electrodynamics in a (2+1)-dimensional Minkowski spacetime. Then the electric field (2.48) in the static case is

$$E_i = -\partial_i A_0. \tag{2.63}$$

Therefore, we see that an electric field is absent in the temporal gauge condition $A_0 = 0$, and the resulting static solution is electrically neutral. Gauss' law in the form (2.62) thus reduces to

$$\nabla^2 A_0 = q|\phi|^2 A_0 \tag{2.64}$$

in the static case. The corresponding (static) Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}|\partial_i A_0|^2 + \frac{1}{2}A_0^2|\phi|^2 + \frac{1}{4}F_{ij}^2 + \frac{1}{2}|D_i\phi|^2 + U(|\phi|^2). \tag{2.65}$$

This brings us to the *Julia–Zee theorem*, which states that finite-energy static solutions in the theory of scalar electrodynamics in a (2+1)-dimensional Minkowski spacetime must be electrically neutral.

More formally, suppose that the field configuration $\{\phi, A\}$ is a solution to the static version of the field equations (2.58) and (2.59) and Gauss' law (2.64) over $\mathbb{R}^2$. Then A_0 must be a constant and so the resulting static solution is automatically electrically neutral and, in particular, if the field ϕ is not identically zero then one requires $A_0 = 0$ everywhere. We will not prove this here, however, the reader is invited to read the paper by Spruck and Yang [30] for a detailed mathematical proof of the theorem. One the main consequences of the Julia–Zee theorem is that it makes it manifest that the static Abelian Higgs theory describes exactly the Ginzburg–Landau model of superconductivity, which is a purely magnetic theory. As we shall see in Section 3, the scalar electrodynamics Lagrangian in the static case becomes exactly that of the Ginzburg–Landau free energy [31].

2.2.1 Scalar electrodynamics on curved spacetimes

In a similar fashion to the simple ungauged theory in Section 2.1.1, we can also easily extend this theory to a spacetime $\mathbb{R} \times M$, where M is a Riemannian manifold equipped with the metric $h_{ij}(\mathbf{x})$ and measure $\sqrt{\det h}\, d^d x$. The Lagrangian density then becomes

$$\mathcal{L} = \frac{1}{2} h^{ij} E_i E_j + \frac{1}{2}\overline{D_0\phi} D_0\phi - \frac{1}{4} h^{ik} h^{jl} F_{ij} F_{kl} - \frac{1}{2} h^{ij} \overline{D_i\phi} D_j\phi - U(|\phi|^2) \tag{2.66}$$

This expression will prove useful later when we consider abelian Higgs vortices on compact hyperbolic surfaces.

In particular, consider a $(2+1)$-dimensional spacetime $\mathbb{R} \times M$, where M is a Riemann surface with a compatible Riemann metric with position-dependent conformal factor $\Omega(x^1, x^2)$. Then the Minkowski metric is

$$(g_{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\Omega & 0 \\ 0 & 0 & -\Omega \end{pmatrix} \tag{2.67}$$

with inverse metric

$$(g^{\mu\nu}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\Omega^{-1} & 0 \\ 0 & 0 & -\Omega^{-1} \end{pmatrix}. \tag{2.68}$$

The scalar electrodynamics action functional is given by

$$S[\phi, A] = \int \int_M \sqrt{g} \left[\frac{1}{2} D_\mu\phi g^{\mu\nu} \overline{D_\nu\phi} - \frac{1}{4} g^{\mu\alpha} g^{\nu\beta} F_{\mu\nu} F_{\alpha\beta} - U(|\phi|^2) \right] d^2x\, dt. \tag{2.69}$$

Taking variations in the fields $\{\bar{\phi}, A_\nu\}$ yields the field equations:

$$\frac{\partial \mathcal{L}}{\partial \bar{\phi}} - D_\mu \left(\frac{\partial \mathcal{L}}{\partial(\overline{D_\mu\phi})} \right) = 0 \quad \Rightarrow D_\mu \left(\sqrt{g} D_\nu\phi g^{\nu\mu} \right) = -\sqrt{g} 2\phi U'(|\phi|^2) \tag{2.70}$$

$$\frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right) = 0 \quad \Rightarrow \partial_\mu \left(\sqrt{g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \right) = -\sqrt{g} g^{\nu\beta} \frac{iq}{2} \left(\bar{\phi} D_\beta\phi - \phi \overline{D_\beta\phi} \right). \tag{2.71}$$

The Minkowski metric $g_{\mu\nu}$ has (modulus of) determinant g, which in this case is $g = \Omega^2$, so that the natural measure is $\sqrt{g} = \Omega$. The second field equation can be written more compactly as

$$\partial_\mu \mathcal{D}^{\mu\nu} = -J^\nu, \tag{2.72}$$

where

$$\mathcal{D}^{\mu\nu} = \sqrt{g} g^{\mu\alpha} g^{\nu\beta} F_{\alpha\beta} \tag{2.73}$$

and

$$J^\nu = \sqrt{g} g^{\nu\beta} \left[\frac{iq}{2} \left(\bar{\phi} \partial_\beta\phi - \phi \partial_\beta\bar{\phi} \right) + q|\phi|^2 A_\beta \right] \tag{2.74}$$

is the Noether current. Focusing on the $\nu = 0$ component of the second field equation, arising from the variation of the A_0 field, the second field equation reduces to

$$\left(\nabla^2 - \Omega q|\phi|^2\right) A_0 = \partial_i \partial_0 A_i + \Omega \frac{iq}{2}\left(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}\right). \tag{2.75}$$

In the static case this is simply

$$\nabla^2 A_0 = \Omega q|\phi|^2 A_0. \tag{2.76}$$

If the Riemann surface M is simply the flat plane $\mathbb{R}^2$, i.e. $\Omega = 1$, then one sees that we readily obtain (2.64).

The corresponding Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\Omega^{-1}|\partial_i A_0|^2 + \frac{1}{2}A_0^2|\phi|^2 + \Omega^{-2}\frac{1}{4}F_{ij}^2 + \frac{1}{2}\Omega^{-1}|D_i\phi|^2 + U(|\phi|^2). \tag{2.77}$$

So, the total energy of the system is therefore

$$E = \int_M \mathcal{H}\,\Omega\, d^2x \tag{2.78}$$

$$= \int_M \left(\frac{1}{2}|\partial_i A_0|^2 + \frac{1}{2}\Omega A_0^2|\phi|^2 + \Omega^{-1}\frac{1}{4}F_{ij}^2 + \frac{1}{2}|D_i\phi|^2 + \Omega U(|\phi|^2)\right) d^2x, \tag{2.79}$$

and the finite energy condition reads

$$\int_M \mathcal{H}\,\Omega\, d^2x < \infty. \tag{2.80}$$

2.2.2 Abelian gauge fields and Chern numbers

Now, we introduce a topological invariant which will prove useful when considering vortices and, in particular, the vortex number $\mathcal{N}$. There is a further generalization of the theory developed in the previous section. We can interpret the field ϕ as a section of a $U(1)$ bundle $\mathcal{B}$ over an unbounded compact Riemann surface M, where the gauge field is a connection 1-form $A = A_1 dx^1 + A_2 dx^2$ on the bundle $\mathcal{B}$. Then the curvature of the connection $F = dA = (\partial_1 A_2 - \partial_2 A_1)\, dx^1 \wedge dx^2$ is the 2-form magnetic field.

We begin by considering a gauge field A in the plane $\mathbb{R}^2$. Then we define the first Chern number c_1 of the complex line bundle over $\mathbb{R}^2$ to be

$$c_1 = \int_{\mathbb{R}^2} C_1 = \frac{1}{2\pi}\int_{\mathbb{R}^2} F = \frac{1}{2\pi}\int_{\mathbb{R}^2} F_{12}\, d^2x, \tag{2.81}$$

where $C_1 = \frac{1}{2\pi}F$ is the first Chern form of A. Hence, it can be seen that the first Chern number of the bundle is a multiple of the total magnetic flux through the plane.

In fact, if one considers a general compact Riemann surface M without boundary, the first Chern number of the bundle $\mathcal{B}$ is

$$c_1 = \frac{1}{2\pi}\int_M F, \tag{2.82}$$

i.e. we still obtain the same constraint on c_1 [2]. There exists a remarkable characterization of the first Chern number c_1, which has powerful implications when considering vortices on M. The number of zeros, counted with multiplicity, of a section ϕ of the $U(1)$ bundle $\mathcal{B}$ is the first Chern number c_1 of the bundle. Vortices are located at the zeros of the field ϕ, so we see that the vortex number $\mathcal{N}$ is the first Chern number c_1 of the Hermitian line bundle $\mathcal{B}$. Therefore, the vortex number $\mathcal{N}$ is a topological invariant and is unaffected by small perturbations of the fields, provided the action is invariant under these perturbations.

2.3 Spontaneous symmetry-breaking, Goldstone bosons and the Higgs mechanism

2.3.1 Spontaneous symmetry-breaking and Goldstone particles

In this section, we want to consider the effect of internal symmetries on possible vacuum configurations. Consider a general Lagrangian in Minkowski spacetime $\mathbb{R} \times \mathbb{R}^d$ which describes n complex scalar fields ϕ, and suppose that the Lagrangian has no explicit time or space dependence. Now let $\mathcal{V}$ be the vacuum manifold of the theory, i.e. the submanifold of $\mathbb{R}^n$ where the potential $U(|\phi|^2)$, and thus the energy, is minimized. Vacua, or ground states, are field configurations $\phi_0 \in \mathcal{V}$ which are constant throughout spacetime, and are the lowest static energy solutions.

Consider the following ungauged Lagrangian density, with $U(1)$ internal symmetry, for a complex spinless field $\phi(x) = \phi_1(x) + i\phi_2(x)$:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \bar{\phi} \partial^\mu \phi - U(|\phi|^2) \quad (\mu = 0, \ldots, d). \tag{2.83}$$

Field configurations which lie on the same orbit of $U(1)$ have the same energy, hence the vacuum isn't required to be unique [2]. If we assume that the potential $U(|\phi|^2)$ is minimized only on a single orbit of $U(1)$ then two possibilities arise. The first possibility is that the orbit consists of a single point. Then the vacuum state is unique, and both the Lagrangian density and the vacuum are invariant under the $U(1)$ symmetry group. This is known as exact symmetry [5]. The second possibility is a non-trivial orbit, which results in a non-invariant vacuum that is not uniquely determined. In this case, we say that the symmetry is broken and this is known as spontaneous symmetry-breaking.

The idea of spontaneous symmetry-breaking, and its implications, is best described by considering the Lagrangian density (2.83) with the Higgs potential

$$U(|\phi|^2) = \frac{\lambda}{8}\left(\phi_0^2 - |\phi|^2\right)^2, \tag{2.84}$$

where ϕ_0 is the ground state and $\lambda > 0$ is the Higgs self-coupling constant (refer to Fig. 1). The trivial case corresponds to the vacuum state $\phi_0 = 0$. Then the minimum of the Higgs potential occurs at $\phi = 0$, and hence the symmetry is unbroken. In this case, our field ϕ describes excitations of two kinds of particles both with the same mass. The symmetric realization of this theory is known as the Wigner-Weyl mode [6]. In this theory with unbroken $U(1)$ symmetry group, the particles have short-range interactions and are massive [2].

We will now consider the case that the vacuum state ϕ_0 is non-zero. One can see instantly from the global phase rotation $\phi(x) \mapsto e^{iq\xi}\phi(x)$ that the symmetry is spontaneously broken. It follows from this that the vacuum must be infinitely degenerate, and the Higgs potential is minimized on the circular orbit $\phi = \phi_0 e^{i\omega}$ for $\omega \in \mathbb{R}$. Consider fluctuations around the vacuum state $\phi = \phi_0 > 0$ and let the field be given by

$$\phi(x) = \phi_0(x) + \phi_1(x) + i\phi_2(x). \tag{2.85}$$

Then we can expand the Lagrangian density (2.83) around the vacuum as

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}(\sqrt{\lambda}\phi_0)^2\phi_1^2 - \frac{\lambda}{8}\left(\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2 + 4\phi_0\phi_1\phi_2^2 + 4\phi_0\phi_1^3\right). \tag{2.86}$$

We see that the mass of the ϕ_1 particles, defined by the coefficient of ϕ_1^2 [5], is $m_1 = \sqrt{\lambda}\phi_0$. Due to the degeneracy, the ϕ_2 excitation about the ground state is massless, and the ϕ_2 particles are referred to as the Goldstone particles. Hence we have shown that spontaneous symmetry-breaking results in massless Goldstone bosons, which have long-range interactions. This encapsulates the Goldstone theorem. The spontaneously broken realization of this theory is known as the Goldstone mode [6].

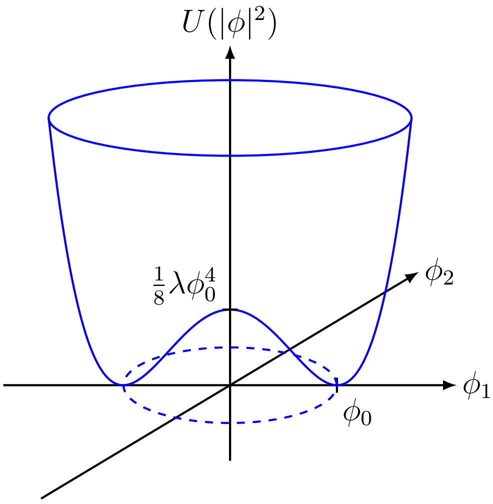


Figure 1: Higgs potential (2.84) for spontaneous breaking of a continuous $U(1)$ symmetry. Oscillations along the bottom of the potential valley correspond to massless Goldstone bosons.

2.3.2 Higgs mechanism

Now we want to consider fluctuations around the vacuum state in an Abelian gauge theory, and see if two different symmetry realizations exist here also. Our aim is to determine whether the fields will be effectively massive or massless. In what follows, it will be shown that when spontaneous symmetry-breaking occurs, the massless Goldstone particles are removed from the system. The second realization of the gauged theory results in a massive real scalar field and the generation of a massive gauge field.

To convey this, consider the scalar electrodynamics Lagrangian (2.52) with the Higgs potential (2.84),

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{\lambda}{8}\left(\phi_0^2 - |\phi|^2\right)^2 \quad (\mu,\nu = 0,\dots,d), \tag{2.87}$$

and consider excitations around the non-zero vacuum state. (The vacuum state here is defined to be $\phi_V = \phi_0 \neq 0$ and $(A_\mu)_V = 0$ [5].) Using the decomposition (2.85), the Higgs potential becomes

$$U(|\phi|^2) = -\frac{1}{2}(\sqrt{\lambda}\phi_0)^2\phi_1^2 - \frac{\lambda}{8}\left(\phi_1^4 + \phi_2^4 + 2\phi_1^2\phi_2^2 + 4\phi_0\phi_1\phi_2^2 + 4\phi_0\phi_1^3\right). \tag{2.88}$$

We see that the field ϕ_1 acquires a mass $m_1 = \sqrt{\lambda}\phi_0$ and again the field ϕ_2 is the massless Goldstone boson. However, the Goldstone boson doesn't actually manifest an independent physical particle [8]. This becomes apparent if one selects a suitable gauge, the unitary gauge, such that the Goldstone boson ϕ_2 disappears. Using the gauge symmetry (2.33), we can choose a suitable gauge transformation such that $\phi(x)$ becomes real-valued for all x, which removes the Goldstone boson from the theory. Then the Lagrangian density (2.87) becomes

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\phi_0^2A_\mu A^\mu - \frac{\lambda}{2}\phi_0^2\phi_1^2 + \mathcal{L}_{\text{int}}, \tag{2.89}$$

where all the off-diagonal interactions are encoded in $\mathcal{L}_{\text{int}}$ [5]. Thus, if the Higgs potential favors a non-zero vacuum $\phi_0 \neq 0$, then we see that the gauge field acquires a mass $m_A = \phi_0$ and couples to the massive real scalar field ϕ_1 with mass $m_1 = \sqrt{\lambda}\phi_0$. The mechanism, by which the spontaneous breaking of a continuous symmetry generates a mass for the gauge boson, is called the Higgs mechanism. These massive gauge bosons are known as Higgs bosons. Unlike the ungauged theory, spontaneous symmetry-breaking does not lead to the presence of a Goldstone boson, but rather to the appearance of a massive Higgs boson. Accordingly, scalar electrodynamics with Lagrangian density (2.87) is often referred to as the Abelian Higgs model, where the complex scalar field ϕ is known as the Higgs field.

The Abelian Higgs model is the main topic of this paper and, in particular, we will discuss the rôle of the Higgs mechanism in superconductivity and its consequences.

2.4 Derrick's theorem

In the previous section, we discussed spontaneous-symmetry breaking and defined the vacuum manifold $\mathcal{V}$ of the theory. Throughout this section, we will restrict our consideration to static field configurations with finite energy. Such field configurations can be classified by their homotopy class where, in particular, the vacuum configuration lies in the trivial homotopy class. If $\phi \in \mathcal{V}$ then it minimizes the energy within the trivial class. This leads one to consider if there exist field configurations in other non-trivial homotopy classes which minimize the energy within their homotopy class. These field configurations are normally stable soliton solutions [2]. There exists a non-existence theorem, known as Derrick's theorem, which determines whether such non-vacuum solutions are possible.

The essence of Derrick's (non-existence) theorem is as follows. Consider a spatial rescaling in $\mathbb{R}^d$, $\mathbf{x} \mapsto \kappa\mathbf{x}$ with $\kappa > 0$. Let ϕ be a critical point of the energy functional $E[\phi]$ and let

$$\phi^{(\kappa)}(\mathbf{x}) = \phi(\kappa\mathbf{x}). \tag{2.90}$$

Further, let

$$e(\kappa) = E[\phi^{(\kappa)}] \tag{2.91}$$

denote the energy of the spatially rescaled field configuration $\phi^{(\kappa)}(\mathbf{x})$ as a function of κ [5]. Derrick's theorem states that if the energy $e(\kappa)$ has no critical point for any arbitrary non-vacuum field configuration $\phi(\mathbf{x})$, then there exist no non-vacuum finite-energy static solutions of the field equations.

From the spatial rescaling (2.90), the gradient operator scales as $\nabla\phi^{(\kappa)}(\mathbf{x}) = \kappa\nabla\phi(\kappa\mathbf{x})$. In order for the covariant derivative of $\phi^{(\kappa)}$ to scale like the field, the 1-form gauge potential A must scale as $A^{(\kappa)}(\mathbf{x}) = \kappa A(\kappa\mathbf{x})$. Thus, the covariant derivative D^A scales as $D^{A^{(\kappa)}}\phi_\kappa(\mathbf{x}) = \kappa D^A\phi(\kappa\mathbf{x})$. We know how the derivative and the gauge potential scale, so it is simple to work out that the 2-form curvature tensor scales as $F^{(\kappa)}(\mathbf{x}) = \kappa^2 F(\kappa\mathbf{x})$. Also, note that the boundary conditions are invariant under the spatial rescaling (2.90), i.e. if $\phi \in \mathcal{V}$ then $\phi^{(\kappa)} \in \mathcal{V}$ and likewise for the particular boundary conditions $D\phi = 0$ and $F = 0$ on the $d-1$-sphere at infinity S^{d-1}_∞. The reason for mentioning the conservation of these particular boundary conditions will become apparent in Section 3.

Recall the Hamiltonian density (2.57). Then the energy functional for scalar electrodynamics is given by the integral of this Hamiltonian density over $\mathbb{R}^d$, i.e

$$\begin{aligned} E[\phi, A] &= \int_{\mathbb{R}^d} \left(\frac{1}{2}|F|^2 + \frac{1}{2}|D\phi|^2 + U(|\phi|^2) \right) d^d x \\ &= E_4 + E_2 + E_0. \end{aligned} \tag{2.92}$$

Under the spatial rescaling (2.90), the energy of the spatially rescaled field configuration $\{\phi^{(\kappa)}, A^{(\kappa)}\}$ is

$$e(\kappa) = \kappa^{4-d}E_4 + \kappa^{2-d}E_2 + \kappa^{-d}E_0. \tag{2.93}$$

For our purposes, we are interested in gauged vortices in two spatial dimensions ($d = 2$) and consider if non-vacuum solutions, which minimize the energy within their topological class, exist. Suppose $\{\phi, A\}$ is a solution to the field equations, where the energy of the theory is given by (2.92). Then the energy of the rescaled field configuration $\{\phi^{(\kappa)}, A^{(\kappa)}\}$ is

$$e(\kappa) = \kappa^2 E_4 + E_2 + \frac{1}{\kappa^2}E_0, \tag{2.94}$$

and so

$$\frac{de}{d\kappa} = 2\kappa E_4 - 2\frac{1}{\kappa^3}E_0. \tag{2.95}$$

We require the derivative (2.95) to vanish at $\kappa = 1$, which yields $E_4 = E_0$. Therefore, we deduce that such non-vacuum solutions exist, provided the contribution from the Maxwell energy E_4 and the energy from the potential energy E_0 are the same.

3 Abelian Higgs vortices on $\mathbb{R}^2$

In 1950, Ginzburg and Landau published a famous paper on the theory of superconductivity, introducing the Ginzburg–Landau model. The gauged Ginzburg–Landau model, or Abelian Higgs model, is known to admit static solitons in two dimensions stabilized by their own magnetic flux. These static solutions are particle-like solutions known as *vortices*, and they exist on a plane or curved Riemann surface M. The relative strength of the forces between vortices are determined by the Higgs self-coupling constant λ. At the critical value $\lambda = 1$ (separating Type I and Type II superconductors), the forces between vortices balance and there exists static multi-vortex solutions. In this case, there exists a lower bound on the energy of a field configuration which is proportional to a topological invariant, the vortex number $\mathcal{N}$ [19].

The key idea is that the dynamics of critically coupled vortices can be approximated by low-velocity motion on the moduli space $\mathcal{M}_\mathcal{N}$ of static solutions. This moduli space $\mathcal{M}_\mathcal{N}$ inherits a natural Kähler structure from the kinetic energy functional of the scalar electrodynamics Lagrangian. Slow-moving vortices are constrained to motion tangent to $\mathcal{M}_\mathcal{N}$, with the trajectories being geodesics on $\mathcal{M}_\mathcal{N}$. The approximate dynamic problem is reduced to finding the geodesics on $\mathcal{M}_\mathcal{N}$ with respect to the Kähler metric induced by the kinetic energy functional. This is known as the geodesic (or moduli space) approximation.

Throughout this section we suppose that space is a two dimensional plane $\mathbb{R}^2$, and we will sometimes make the identification $\mathbb{R}^2 \cong \mathbb{C}$. On the complex plane $\mathbb{C}$ we shall denote a spatial point by $z = x^1 + ix^2$. Note that throughout the rest of this paper we have normalized the electric charge to unity, i.e. $q = 1$. Our aim is to investigate the dynamics of critically coupled vortices on $\mathbb{R}^2$, and determine the form of the metric on the moduli space $\mathcal{M}_\mathcal{N}$. The further generalization, where we discuss critically coupled vortices on compact Riemann surfaces, is covered in Section 4. In this case, we are able to find surfaces which admit integrable vortices. However, here we shall restrict our focus to vortices on the flat plane $\mathbb{R}^2$. (It is worth noting that the static solutions on $\mathbb{R}^2$, although known to exist, are not known in closed form [17].) We begin by building on the formalism introduced in Sections 2.2-2.3, and our starting point is the (2+1)-dimensional Abelian Higgs model.

3.1 Abelian Higgs model

The Abelian Higgs model comprises a complex scalar Higgs field $\phi(x) = \phi_1(x) + i\phi_2(x)$ coupled to a $U(1)$ gauge field $A_\mu(x)$, along with a symmetry-breaking Higgs potential. Alternatively, we can interpret the Higgs field ϕ as a section of a Hermitian line bundle over $\mathbb{R}^2$, where A_μ is a connection on the line bundle and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ $(\mu, \nu = 0, 1, 2)$ is the curvature [3, 19]. We will consider the symmetry-breaking Higgs potential given by

$$U(|\phi|^2) = \frac{\lambda}{8}\left(m^2 - |\phi|^2\right)^2, \tag{3.1}$$

where $m > 0$ is the vacuum state and, for a stable theory, we require $\lambda > 0$. As before, λ is the Higgs self-coupling constant. The vacuum manifold of this theory is the circle $\mathcal{V} = S^1_m$, i.e. $|\phi| = m$, with the non-trivial homotopy group $\pi_1(\mathcal{V}) = \mathbb{Z}$.

The Lagrangian for the (2+1)-dimensional Abelian Higgs model is given by the Lagrangian density for scalar electrodynamics (2.52) with the Higgs potential (3.1):

$$L[\phi, A] = \int_{\mathbb{R}^2} \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\overline{D_\mu\phi}D^\mu\phi - \frac{\lambda}{8}\left(m^2 - |\phi|^2\right)^2 \right) d^2x. \tag{3.2}$$

Expanding the Lagrangian (3.2) around the vacuum state m, as we did in Section 2.3.2, we see that the Higgs field acquires a mass $m_1 = \sqrt{\lambda}m$. A consequence of the spontaneous symmetry-breaking is that the gauge field acquires a mass $m_A = m$. The potential energy in the Abelian Higgs model is

$$V[\phi, A] = \frac{1}{2}\int_{\mathbb{R}^2} \left(F_{12}^2 + \overline{D_i\phi} D_i\phi + \frac{\lambda}{4}\left(m^2 - |\phi|^2\right)^2 \right) d^2x, \tag{3.3}$$

where $F_{12} = \partial_1 A_2 - \partial_2 A_1 (= -B_3)$ is the magnetic field in the $-x^3$ direction. This is is the two-dimensional analogue of the potential (2.55). The Abelian Higgs Lagrangian (3.2), and hence the potential energy (3.3), is invariant under the local gauge transformations

$$\phi(x) \mapsto e^{i\xi(x)}\phi(x) \tag{3.4}$$
$$A_\mu(x) \mapsto A_\mu(x) + \partial_\mu \xi(x). \tag{3.5}$$

One can readily obtain the field equations for the *static* Abelian Higgs model, i.e. $E = V$, by considering variations of the fields $\{\bar{\phi}, A_i\}$. The corresponding field equations are then

$$D_i D_i \phi + \frac{\lambda}{2}\left(m^2 - |\phi|^2\right)\phi = 0 \tag{3.6}$$
$$\varepsilon_{ij}\partial_j F_{12} + \frac{i}{2}\left(\bar{\phi} D_i\phi - \phi\overline{D_i\phi}\right) = 0. \tag{3.7}$$

Note that (3.7) is the two-dimensional analogue of Ampère's law $\nabla \times \mathbf{B} = \mathbf{j}$ [2], so we can interpret

$$J^i = \frac{i}{2}\left(\bar{\phi} D_i\phi - \phi\overline{D_i\phi}\right) \tag{3.8}$$

as the electric current in the plane, which is consistent with (2.60). The static Abelian Higgs Lagrangian is exactly that of the Ginzburg–Landau free energy in the Ginzburg–Landau theory of superconductors [15].

In the Abelian Higgs model the vacuum is unique (up to gauge equivalence), and the Ginzburg–Landau energy is minimized if the following conditions hold over the whole plane: $F_{12} = 0, D_i\phi = 0$ and $|\phi| = m$. The first condition, $F_{12} = \partial_1 A_2 - \partial_2 A_1 = 0$, requires the gauge field A_i to be a pure gauge, i.e.

$$A_i(x) = \partial_i \xi(x). \tag{3.9}$$

The last condition, $|\phi| = m$, requires the Higgs field to be of the form $\phi(\mathbf{x}) = me^{i\alpha(\mathbf{x})}$. If the covariant derivative is everywhere zero then, using the pure gauge (3.9), we have

$$D_i\phi = ime^{i\alpha(\mathbf{x})}\left(\partial_i\alpha(\mathbf{x}) - \partial_i\xi(\mathbf{x})\right) = 0. \tag{3.10}$$

Therefore, the vacuum state of the theory is of the form $\{\phi, A_i\} = \{me^{i\alpha(\mathbf{x})}, \partial_i\alpha(\mathbf{x})\}$. But we can gauge transform the vacuum by $e^{-i\alpha(\mathbf{x})}$, so that it becomes the simple vacuum $\{\phi, A_i\} = \{m, 0\}$, which is unique [2].

From Derrick's theorem, there can exist non-vacuum solutions with finite energy in the Abelian Higgs model. Recall from Section 2.4 that, for $d = 2$, the field configuration $\{\phi, A\}$ is a solution to the field equations provided $E_4 = E_0$, where

$$E_4 = \frac{1}{2}\int_{\mathbb{R}^2} F_{12}^2\, d^2x, \quad E_0 = \frac{\lambda}{8}\int_{\mathbb{R}^2}\left(m^2 - |\phi|^2\right)^2 d^2x. \tag{3.11}$$

In other words, non-vacuum static solutions with finite energy exist provided there is an equal contribution, to the total energy, from the Maxwell energy E_4 and the energy from the Higgs potential E_0.

3.2 Topology in the Abelian Higgs model

For this section, it will be convenient to work in polar coordinates (ρ, θ). Transforming to polar coordinates (ρ, θ), the 1-form connection is coordinate invariant and can be written as

$$A = A_1 dx^1 + A_2 dx^2 = A_\rho d\rho + A_\theta d\theta, \tag{3.12}$$

and the magnetic field in the plane becomes

$$F_{\rho\theta} = (\partial_\rho A_\theta - \partial_\theta A_\rho) = \rho F_{12}. \tag{3.13}$$

The covariant derivatives in this coordinate system are $D_\rho \phi = \partial_\rho \phi - iA_\rho \phi$ and $D_\theta \phi = \partial_\theta \phi - iA_\theta \phi$. Thus, using the Lagrangian density (2.66) where the Riemann manifold M is the plane $\mathbb{R}^2$ equipped with the metric $ds^2 = d\rho^2 + \rho^2 d\theta^2$, the Ginzburg–Landau energy becomes

$$V[\phi, A] = \frac{1}{2}\int_0^\infty \int_0^{2\pi} \left(\frac{1}{\rho^2} F_{\rho\theta}^2 + \overline{D_\rho \phi} D_\rho \phi + \frac{1}{\rho^2} \overline{D_\theta \phi} D_\theta \phi + \frac{\lambda}{4}\left(m^2 - |\phi|^2\right)^2 \right) \rho \, d\rho \, d\theta. \tag{3.14}$$

Our aim is to show that the Higgs field ϕ is a pure phase on the circle at infinity S^1_∞, and that the gauge field A is a pure gauge on S^1_∞. We will begin by deducing a limiting form of ϕ on S^1_∞ and show that this is indeed pure phase. Finiteness of the Ginzurg-Landau energy implies the boundary conditions $|\phi| \to m$ and $D_\rho \phi \to 0$ as $|\mathbf{x}| \to \infty$ [16]. Consider a fixed radial line $0 \leq \rho < \infty$ [2] and let $\rho \to \infty$, so that $\phi \to m e^{i\alpha}$. Then, asymptotically, we have

$$D_\rho \phi = ime^{i\alpha}(\partial_\rho \alpha - A_\rho) = 0, \tag{3.15}$$

which yields $A_\rho = \partial_\rho \alpha$. Now, let us transform to the radial gauge $A_\rho = 0$. In the radial gauge, the vanishing of the covariant derivative $D_\rho \phi$ at spatial infinity requires $\partial_\rho \phi \to 0$ as $|\mathbf{x}| \to \infty$ along a fixed radial line. Hence the limiting form of ϕ on S^1_∞ is $\lim_{\rho \to \infty} \phi(\rho, \theta) = \phi^\infty(\theta)$. Thus, we deduce that the Higgs field ϕ is a pure phase on S^1_∞, and is given by

$$\phi^\infty(\theta) = me^{i\alpha^\infty(\theta)}. \tag{3.16}$$

Now we will show that the gauge field A is a pure gauge on S^1_∞. Finite energy also imposes the boundary conditions $F_{\rho\theta} \to 0$ and $D_\theta \phi \to 0$ as $|\mathbf{x}| \to \infty$. In the radial gauge $A_\rho = 0$ the magnetic field in the plane is $F_{\rho\theta} = \partial_\rho A_\theta = 0$. Thus, the gauge field component A_θ has the limiting form $\lim_{\rho \to \infty} A_\theta(\rho, \theta) = A_\theta^\infty(\theta)$. Now the vanishing of the covariant derivative $D_\theta \phi$ at spatial infinity yields

$$D_\theta \phi^\infty = \partial_\theta \phi^\infty - iA_\theta^\infty \phi^\infty = ime^{i\alpha^\infty}\left(\partial_\theta \alpha^\infty - A_\theta^\infty\right) = 0. \tag{3.17}$$

Therefore, we see that the gauge field is a pure gauge on S^1_∞, given by

$$A_\theta^\infty = \partial_\theta \alpha^\infty. \tag{3.18}$$

The Higgs field at infinity ϕ^∞ is a map from the circle at infinity S^1_∞ to the vacuum manifold $\mathcal{V} = S^1_m$,

$$\phi^\infty : S^1_\infty \mapsto S^1_m. \tag{3.19}$$

This map has an associated topological degree, or winding number, $\deg(\phi^\infty) = \mathcal{N} \in \mathbb{Z}$ given by

$$\mathcal{N} = \frac{1}{2\pi}\int_0^{2\pi} \partial_\theta \alpha^\infty(\theta) \, d\theta. \tag{3.20}$$

Now recall from Section 2.2.2 that the first Chern number c_1 of a $U(1)$ bundle $\mathcal{B}$ over $\mathbb{R}^2$ is given by (2.81). On the circle at infinity S^1_∞, the gauge field is given by (3.18). Then, using Stoke's theorem for differential forms, the first Chern number c_1 can be expressed as a line integral along the circle at infinity [2],

$$c_1 = \frac{1}{2\pi}\int_{\mathbb{R}^2} F = \frac{1}{2\pi}\oint_{S^1_\infty} A = \frac{1}{2\pi}\int_0^{2\pi} \partial_\theta \alpha^\infty(\theta)\, d\theta. \tag{3.21}$$

Thus, one easily sees that the winding number $\mathcal{N}$ is the first Chern number c_1. As we discussed in Section 2.2.2, the first Chern number c_1 is the vortex number, i.e. the number of zeros of the Higgs field ϕ counted with multiplicity. It follows from this that the winding number $\mathcal{N}$ is the vortex number.

3.3 Critically coupled vortices

The Higgs self-coupling constant λ determines the relative strengths of the forces between vortices. As a result of the spontaneous symmetry-breaking, the fields are massive and the vortices have short-range interactions. In the Type I superconductor regime, $\lambda < 1$, the scalar forces prevail and the vortices are attractive [25]. Whereas, in the Type II superconductor regime, $\lambda > 1$, the magnetostatic forces are dominant [23] and the vortices repel one another. At the critical coupling value $\lambda = 1$ the forces between vortices balance, and there exists static multi-vortex solutions corresponding to vortices located arbitrarily in the plane. In this case, we will see that the static energy of a field configuration is bounded below by the vortex number $\mathcal{N}$, which is a topological invariant.

Throughout the rest of this section, let us set the vacuum expectation value $m = 1$, so that the vacuum manifold is now the unit circle $\mathcal{V} = S^1$. So the Higgs field at infinity ϕ^∞ is the map $\phi^\infty : S^1_\infty \mapsto S^1$ with integer winding number $\mathcal{N} = \deg(\phi^\infty)$. The mass of the Higgs field ϕ is then $m_1 = \sqrt{\lambda}$ and the gauge field mass is $m_A = 1$. Hence, the Ginzburg–Landau energy (3.3) simply becomes

$$E[\phi, A] = V = \frac{1}{2}\int_{\mathbb{R}^2}\left(F_{12}^2 + \overline{D_i\phi}D_i\phi + \frac{\lambda}{4}(1-|\phi|^2)^2\right) d^2x. \tag{3.22}$$

This can be rearranged into the Bogomolny form by integrating by parts,

$$\begin{aligned} E[\phi, A] = &\frac{1}{2}\int_{\mathbb{R}^2}\left\{\left[F_{12} - \frac{1}{2}\left(1-|\phi|^2\right)\right]^2 + \left(\overline{D_1\phi} - i\overline{D_2\phi}\right)\left(D_1\phi + iD_2\phi\right)\right\} d^2x \\ &+ \frac{1}{2}\int_{\mathbb{R}^2}\left\{i\left(\overline{D_2\phi}D_1\phi - \overline{D_2\phi}D_1\phi\right) + F_{12}\left(1-|\phi|^2\right)\right\} d^2x + \frac{\lambda-1}{4}\int_{\mathbb{R}^2}\left(1-|\phi|^2\right)^2 d^2x. \end{aligned} \tag{3.23}$$

We will focus on critically coupled vortices ($\lambda = 1$), so that the energy decomposes into positive definite terms [13, 25]. This also forces the Higgs mass and the gauge boson mass to be equal to unity. Using the commutation relation (2.41), the third term can be written as

$$\begin{aligned} \overline{D_2\phi}D_1\phi - \overline{D_2\phi}D_1\phi &= \left(\partial_2\bar{\phi} + iA_2\bar{\phi}\right)D_1\phi - \left(\partial_1\bar{\phi} + iA_1\bar{\phi}\right)D_2\phi \\ &= \partial_2\left(\bar{\phi}D_1\phi\right) - \partial_1\left(\bar{\phi}D_2\phi\right) + [D_1, D_2]\,|\phi|^2 \\ &= \partial_2\left(\bar{\phi}D_1\phi\right) - \partial_1\left(\bar{\phi}D_2\phi\right) - iF_{12}|\phi|^2. \end{aligned} \tag{3.24}$$

Thus, the static Ginzburg–Landau energy is now

$$\begin{aligned} E[\phi, A] = &\frac{1}{2}\int_{\mathbb{R}^2} F_{12}\, d^2x + \frac{1}{2}\int_{\mathbb{R}^2}\left\{\left[F_{12} - \frac{1}{2}\left(1-|\phi|^2\right)\right]^2 + \left(\overline{D_1\phi} - i\overline{D_2\phi}\right)\left(D_1\phi + iD_2\phi\right)\right\} d^2x \\ &- \frac{i}{2}\int_{\mathbb{R}^2}\left\{\partial_1\left(\bar{\phi}D_2\phi\right) - \partial_2\left(\bar{\phi}D_1\phi\right)\right\} d^2x. \end{aligned} \tag{3.25}$$

One readily sees that the first term is proportional to the first Chern number c_1 via (2.81) – it is exactly π times the first Chern number c_1. As we already discussed in Section 3.2, the first Chern number c_1 is the winding number or vortex number $\mathcal{N}$. So the first term is in fact π times the vortex number $\mathcal{N}$. Now, we can express the last intergral as a line integral around the circle at infinity S^1_∞,

$$\begin{aligned}\int_{\mathbb{R}^2} \left\{\partial_1\left(\bar{\phi}D_2\phi\right) - \partial_2\left(\bar{\phi}D_1\phi\right)\right\}\, d^2x &= \int_{\mathbb{R}^2} \nabla \times \left(\bar{\phi}D_2\phi, \bar{\phi}D_1\phi\right)\, d^2x \\ &= \oint_{S^1_\infty} \left(\bar{\phi}D_2\phi, \bar{\phi}D_1\phi\right)\cdot d\mathbf{x},\end{aligned} \tag{3.26}$$

which vanishes since the boundary conditions are now $|\phi| \to 1$ and $D_i\phi \to 0$ as $|\mathbf{x}| \to \infty$. Therefore, the static energy may finally be written as

$$E[\phi, A] = \pi\mathcal{N} + \frac{1}{2}\int_{\mathbb{R}^2}\left\{\left[F_{12} - \frac{1}{2}\left(1-|\phi|^2\right)\right]^2 + \left(\overline{D_1\phi} - i\overline{D_2\phi}\right)\left(D_1\phi + iD_2\phi\right)\right\} d^2x. \tag{3.27}$$

Thus, in general, the static Ginzburg–Landau energy (3.27) has a lower bound, known as the *Bogomolny bound*:

$$E \geq \pi|\mathcal{N}|. \tag{3.28}$$

From here on, we are only going to be considering the case of positive vortex number $\mathcal{N} > 0$, however, the negative vortex number $\mathcal{N} < 0$ case is analogous [19]. Solutions corresponding to $\mathcal{N} > 0$ are called vortices, whereas for $\mathcal{N} < 0$ they are referred to as anti-vortices. The Bogomolny bound is saturated, i.e. $E = \pi\mathcal{N}$, if and only if the Higgs field ϕ satisfies the Bogomolny equations[1]:

$$D_1\phi + iD_2\phi = 0 \tag{3.29}$$

$$F_{12} - \frac{1}{2}\left(1-|\phi|^2\right) = 0. \tag{3.30}$$

Field configurations which satisfy the first order Bogomolny equations (3.29) and (3.30) minimze the static energy within their topological class, and thus are automatically solutions of the second order static field equations (3.6) and (3.7). Jaffe and Taubes [3] showed that all the solutions satisfying the static field equations also satisfy the Bogomolny equations. Thus, it is sufficient to study the Bogomolny equations in order to find solutions to the full static field equations.

We briefly digress to discuss the relation between the zeros of the Higgs field ϕ and its rôle in the theory of superconductivity. The complex scalar Higgs field ϕ plays the rôle of an order parameter, which gives rise to the density distribution of superconducting electron pairs, known as Cooper pairs [5]. In particular, the Higgs field ϕ characterizes two phases of a solid: superconductivity and normal states. If the order parameter ϕ is non-zero then it indicates the presence of Cooper pairs and, hence, superconductivity. Whereas, if $\phi = 0$ then it indicates the absence of Cooper pairs and the presence of normal states. Consequently, if the order parameter ϕ is such that it is vanishing at some points and non-vanishing elsewhere, then there exists a mixed state of superconducting and normal states. When a superconductor transitions to its superconducting phase, then the magnetic field is unable to penetrate the superconductor – this is known as the *Meissner effect*. Therefore, in this mixed state, the magnetic field F_{12} attains its local maxima at the positions where the order parameter ϕ vanishes, i.e. the zeros of the Higgs field ϕ. Similarly, $|D_i\phi|^2$ also assumes its local maxima at the zeros of the Higgs field ϕ. Thus, the Hamiltonian density of the Ginzburg–Landau free energy (3.22),

$$\mathcal{H}[\phi, A] = \frac{1}{2}F_{12}^2 + \frac{1}{2}\overline{D_i\phi}D_i\phi + \frac{\lambda}{8}(1-|\phi|^2)^2, \tag{3.31}$$

also attains its local maxima at the zeros of the Higgs field ϕ. Essentially we have observed that energy, and equivalently mass, are concentrated around the zeros of the Higgs field ϕ. It should now be clear that we can identify these static soliton solutions with particles.

[1]It is easy to check that the second Bogomolny equation (3.30) is in accordance with Derrick's theorem, $E_4 = E_0$.

Another distinct result of Jaffe and Taubes [3] is that the multiplicities of the zeros of the Higgs field ϕ are all positive, provided ϕ satisfies the Bogomolny equations. Hence, the sum of the multiplicities is the vortex number $\mathcal{N}$. Thus, we see that solutions of the static field equations correspond to $\mathcal{N}$ static vortices, located at the zeros of the Higgs field ϕ.

Using the identification $\mathbb{R}^2 \cong \mathbb{C}$, let us introduce the local complex coordinate $z = x^1 + ix^2$, and the following complex derivatives

$$\partial_z = \frac{1}{2}(\partial_1 - i\partial_2), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2). \tag{3.32}$$

The connection 1-form is coordinate invariant and is given by $A = A_z dz + A_{\bar{z}} d\bar{z}$, where the gauge potential components are

$$A_z = \frac{1}{2}(A_1 - iA_2), \quad A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2). \tag{3.33}$$

The curvature of the connection is the 2-form $F = F_{z\bar{z}}\, dz \wedge d\bar{z}$, where

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = \frac{i}{2} F_{12}. \tag{3.34}$$

This enables us to rewrite the Bogomolny equations (3.29) and (3.30) in terms of the local complex coordinate z (and its complex conjugate $\bar{z}$) as

$$D_{\bar{z}}\phi = \partial_{\bar{z}}\phi - iA_{\bar{z}}\phi = 0 \tag{3.35}$$

$$F_{z\bar{z}} = \frac{i}{4}(1 - |\phi|^2). \tag{3.36}$$

In this form, it is easily seen that the first Bogomolny equation (3.35) is a covariant generalization of the Cauchy-Riemann equations. Is also allows us to eliminate the gauge field, via $A_{\bar{z}} = -i\partial_{\bar{z}} \log \phi$. Since the gauge group $U(1)$ is Abelian, A_z is simply the complex conjugate of $A_{\bar{z}}$, i.e. $A_z = i\partial_z \log \bar{\phi}$. Let us express the Higgs field ϕ in terms of a gauge invariant quantity h by $h = \log |\phi|^2$. The boundary condition $|\phi| \to 1$ as $|\mathbf{x}| \to \infty$ implies that $h \to 0$ at spatial infinity. Hence, the magnetic field $F_{z\bar{z}}$ becomes

$$F_{z\bar{z}} = \partial_z A_{\bar{z}} - \partial_{\bar{z}} A_z = -i\partial_z\partial_{\bar{z}} \log |\phi|^2 = -i\partial_z\partial_{\bar{z}} h. \tag{3.37}$$

Away from the zeros of the Higgs field ϕ, where h has logarithmic singularities [32], the second Bogomolny equation (3.36) is now

$$4\partial_z\partial_{\bar{z}} h + 1 - e^h = 0. \tag{3.38}$$

To allow for singularities, we can supplement this by δ-function sources to obtain the gauge invariant Taubes equation [16]

$$4\partial_z\partial_{\bar{z}} h + 1 - e^h = 4\pi \sum_{r=1}^{\mathcal{N}} \delta^2(z - Z_r), \tag{3.39}$$

where $\{Z_r : 1 \leq r \leq \mathcal{N}\}$ are the vortex centres in $\mathbb{C}$, i.e. the (simple) zeros of the Higgs field ϕ. Close to a vortex centre Z_r, the δ-function sources enforce that $h \sim 2\log|z - Z_r|$ and so $|\phi| \sim |z - Z_r|$ in the neighbourhood of the simple zero Z_r. Away from the zeros, the Higgs fields ϕ rapidly approaches its asymptotic value. We see that, for well-separated vortices, we can approximate an $\mathcal{N}$-vortex solution as a superposition of $\mathcal{N}$ charge-1 vortices, each of which we can regard as a single particle of mass π carrying a magnetic flux 2π [23]. Note that the magnetic flux and the energy density are concentrated about the vortex centres. Thus, solutions of Taubes' equation (3.39) can reasonably be labelled as multi-vortex solutions of the Bogomolny equations (3.35) and (3.36) [25].

Taubes' existence theorem [19] states that there exists a unique solution $\{\phi, A\}$ to the Bogomolny equations (3.35) and (3.36), determined by an arbitrary point $(Z_1, \ldots, Z_{\mathcal{N}}) \in \mathbb{C} \times \cdots \times \mathbb{C} = \mathbb{C}^{\mathcal{N}}$. This is provided the field configuration $\{\phi, A\}$ satisfies the boundary conditions $|\phi| \to 1$ and $D_i\phi \to 0$ as $|z| \to \infty$, and the further condition

$$\{z \in \mathbb{C} : \phi(z) = 0\} = \bigcup_{r=1}^{\mathcal{N}} \{Z_r\}. \tag{3.40}$$

The $\mathcal{N}$-vortex moduli space $\mathcal{M}_{\mathcal{N}}$ is the space of smooth solutions (modulo gauge equivalence [23]) to the Bogomolny equations with winding number $\mathcal{N}$. By Taubes' existence theorem, we can uniquely determine each solution by choosing $\mathcal{N}$ unordered arbitrary points in $\mathbb{C}$, provided they satisfy the condition (3.40). The moduli space $\mathcal{M}_{\mathcal{N}}$ is a $2\mathcal{N}$-dimensional manifold, in particular, the moduli space is $\mathbb{C}^{\mathcal{N}}/S_{\mathcal{N}}$, where $S_{\mathcal{N}}$ is the permutation group on $\mathcal{N}$ objects [2]. This is due to solutions of Taubes' equation (3.39) being invariant under interchanges of vortex centres [25].

To obtain good global coordinates on the moduli space $\mathcal{M}_{\mathcal{N}}$, we define the degree $\mathcal{N}$ complex polynomial [23, 25]

$$\begin{aligned} P(z) &= P_0 + P_1 z + \cdots + P_{\mathcal{N}-1} z^{\mathcal{N}-1} + z^{\mathcal{N}} \\ &= \prod_{r=1}^{\mathcal{N}} (z - Z_r), \end{aligned} \tag{3.41}$$

where the vortex centres $\{Z_r : 1 \leq r \leq \mathcal{N}\}$ are the roots of the monic polynomial. The set of coefficients $\{P_r : 0 \leq r \leq \mathcal{N} - 1\}$ provide good global coordinates on the moduli space $\mathcal{M}_{\mathcal{N}}$, and we see that the moduli space is the topologically trivial space $\mathcal{M}_{\mathcal{N}} = \mathbb{C}^{\mathcal{N}}$ [23], which is diffeomorphic to $\mathbb{C}^{\mathcal{N}}/S_{\mathcal{N}}$ [2]. Thus, the moduli space $\mathcal{M}_{\mathcal{N}}$ inherits the natural complex manifold structure of $\mathbb{C}^{\mathcal{N}}$.

3.3.1 Dynamics of critically coupled vortices

Before we go on to investigate the moduli space $\mathcal{M}_{\mathcal{N}}$ and the form of the metric on $\mathcal{M}_{\mathcal{N}}$, we briefly discuss the scattering behaviour of vortices in head-on collisions. As we are interested in low-energy phenomena, our discussion will restrict to non-relativistic collisions where negligible radiation is produced.

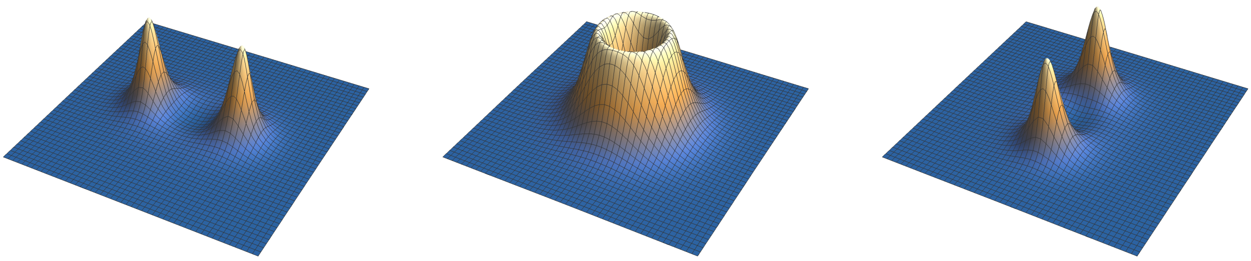
Consider the head-on collision of two critically coupled ($\lambda = 1$) vortices. Low-velocity collisions are approximately adiabatic, so the difference in energy before and after the collision is negligible. Initially they are well-separated, so we can regard each vortice as an individual particle of mass π. Suppose the vortices collide at the centre of mass of the system, they subsequently scatter at 90° relative to the incoming vortices. This observed phenomenon is referred to as right-angle scattering, and has been observed for all coupling values λ studied [11]. The right-angle scattering of two critically coupled vortices is shown in Fig. 2.

There is a generalization of the right-angle scattering of two vortices in a head-on collision to $\mathcal{N}$ vortices. Consider $\mathcal{N}$ non-relativistic classically indistinguishable vortices with $C_{\mathcal{N}}$ symmetry, which collide simultaneously at the centre of mass of the system. The vortices emerge from the collision rotated by $\pi/\mathcal{N}$ relative to the incoming one [2]. Energy density plots of $\pi/4$ scattering of four vortices with C_4 symmetry are shown in Fig. 3.

For non-relativistic critically coupled vortices, we can model the vortex dynamics by geodesic motion on the moduli space $\mathcal{M}_{\mathcal{N}}$ of static solutions [27]. This brings us to the discussion of the *geodesic approximation*.

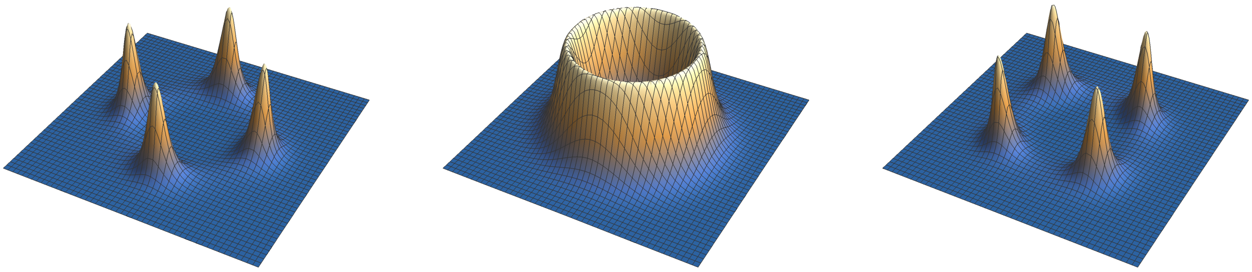
3.4 Geodesic approximation

Our aim is to motivate the geodesic approximation of low-velocity scattering of vortices at critical coupling. Let $\mathcal{A}$ be the set of finite energy spatial field configurations $a = \{\phi(\mathbf{x}), A_i(\mathbf{x})\}$ in $\mathbb{R}^2$, and



(a) (b) (c)

Figure 2: Energy density plots of two critically coupled vortices during a head-on collision, illustrating the right-angle scattering phenomena.



(a) (b) (c)

Figure 3: Energy density plots of four vortices with C_4 symmetry in a head-on collision, illustrating the $\pi/4$ scattering.

$\mathcal{G} = \left\{e^{i\xi(\mathbf{x})}\right\}$ be the group of gauge transformations over $\mathbb{R}^2$ [10]. Then the true configuration space is the quotient $\mathcal{C} = \mathcal{A}/\mathcal{G}$, which is a manifold since the gauge transformations tend to the identity at spatial infinity [12]. The kinetic energy is well-defined on $\mathcal{A}$ [2], and there exists a natural metric l on $\mathcal{A}$, given by [10]

$$l(\partial_0 a, \partial_0 a) = \frac{1}{2}\int_{\mathbb{R}^2} \left(\partial_0 A_i \partial_0 A_i + \partial_0\bar{\phi}\partial_0\phi\right)\, d^2x. \tag{3.42}$$

Taking the inner product of $\partial_0 a$ with an infinitesimal gauge transformation $\partial_0\tau = \{i\Lambda\phi, \partial_i\Lambda\}$ yields [23]

$$\begin{aligned} l(\partial_0 a, \partial_0\tau) &= \frac{1}{2}\int_{\mathbb{R}^2}\left(\partial_0 A_i\partial_i\Lambda - \frac{i\Lambda}{2}(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi})\right)\, d^2x \\ &= -\frac{1}{2}\int_{\mathbb{R}^2}\left(\partial_i\partial_0 A_i + \frac{i}{2}(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi})\right)\Lambda\, d^2x, \end{aligned} \tag{3.43}$$

where we have used integration by parts on the first term. However, working in the $A_0 = 0$ gauge, Gauss' law (2.62) becomes

$$\partial_i\partial_0 A_i + \frac{i}{2}(\bar{\phi}\partial_0\phi - \phi\partial_0\bar{\phi}) = 0. \tag{3.44}$$

Therefore, we see that Gauss' law (3.44) imposes that the generalized velocity $\partial_0 a$ is orthogonal in the natural metric to the gauge group orbits through a [23]. By projecting the velocity, the metric (3.42) is well defined on the true configuration space $\mathcal{C}$ – it is the kinetic energy functional (2.54) over $\mathbb{R}^2$:

$$\begin{aligned} T[\phi, A] &= \int_{\mathbb{R}^2} \left(\frac{1}{2} E_i E_i + \frac{1}{2} \overline{D_0 \phi} D_0 \phi \right) d^2x \\ &= \frac{1}{2} \int_{\mathbb{R}^2} \left(\partial_0 A_i \partial_0 A_i + \partial_0 \bar{\phi} \partial_0 \phi \right) d^2x. \end{aligned} \tag{3.45}$$

Since the Ginzburg–Landau energy (3.22) is trivially gauge invariant, and thus well defined on $\mathcal{C}$, we can project the Lagrangian (3.2) (at critical coupling, $\lambda = 1$) to an action on $\mathcal{C}$ [12]. Therefore, the resulting dynamic problem can be interpreted as motion on $\mathcal{C}$ with the metric defined by the kinetic energy (3.45), and the potential energy (3.22) [23].

The geodesic approximation is a powerful method for investigating the low-energy dynamics and scattering of vortices. At low velocities, the collisions of vortices can be treated adiabatically via the geodesic approximation. The main idea is that these slowly-moving vortices are restricted to motion tangent to the moduli space of static solutions $\mathcal{M}_{\mathcal{N}}$, where the evolution is governed by the geodesic trajectories on $\mathcal{M}_{\mathcal{N}}$. The kinetic energy functional (3.45) induces the metric on the moduli space $\mathcal{M}_{\mathcal{N}}$, which, as we will see in Section 3.6, is Kähler. Thus, the dynamics of the vortices can be accurately modelled by geodesic motion on $\mathcal{M}_{\mathcal{N}}$ with respect to the Kähler metric induced by the kinetic energy – this is the geodesic approximation[2].

3.5 Moduli space dynamics

It will be convenient for us to use the set of *arbitrary ordered* vortex centres $\{Z_r : 1 \leq r \leq \mathcal{N}\}$ as the coordinates of the $2\mathcal{N}$-dimensional moduli space $\mathcal{M}_{\mathcal{N}} = \mathbb{C}^{\mathcal{N}}$. Note that $\mathbb{C}^{\mathcal{N}} = \{Z_r : 1 \leq r \leq \mathcal{N}\}$, where the set is ordered, is a branched covering of $\mathcal{M}_{\mathcal{N}}$ [23]. These coordinates are locally well defined for *separated* vortices on $\mathcal{M}_{\mathcal{N}}$. Let $\Delta_{\mathcal{N}}$ be the $(2\mathcal{N} - 2)$-dimensional subspace, where some or all vortices coincide [2]. Then the coordinates breakdown on the subspace $\Delta_{\mathcal{N}}$. If we imagine the vortices are described by point particles, the resulting metric is flat and well defined on $\mathbb{C}^{\mathcal{N}} = \{Z_r : 1 \leq r \leq \mathcal{N}\}$. However, this metric is not well defined on the moduli space $\mathcal{M}_{\mathcal{N}}$, where it has conical singularities on $\Delta_{\mathcal{N}}$ [23]. As the vortices have short-range interactions, this 'point-particle' metric and the vortex metric will agree asymptotically [23], which can easily be seen when one considers the 2-vortex moduli space $\mathcal{M}_2$ (see Fig. 5).

As previously discussed, the dynamics of critically coupled vortices reduces to geodesic motion on the moduli space $\mathcal{M}_{\mathcal{N}}$. In this case, the potential energy is minimized and the Lagrangian becomes purely kinetic. This kinetic energy induces a metric on $\mathcal{M}_{\mathcal{N}}$, and calculating the form of the moduli space metric will be the topic of the next section. Consider the subspace $\mathcal{M}_{\mathcal{N}} \backslash \Delta_{\mathcal{N}}$ with distinct coordinates $\{Z_r : 1 \leq r \leq \mathcal{N}\}$, this can be extended to $\mathcal{M}_{\mathcal{N}}$ by continuity [23]. In terms of the coordinates, the metric is [2, 23]

$$ds^2 = \sum_{r,s=1}^{\mathcal{N}} \left(g_{rs}\, dZ_r dZ_s + g_{r\bar{s}}\, dZ_r d\bar{Z}_s + g_{\bar{r}\bar{s}}\, d\bar{Z}_r d\bar{Z}_s \right). \tag{3.46}$$

where the metric coefficients g_{rs}, $g_{r\bar{s}}$ and $g_{\bar{r}\bar{s}}$ are dependent on the coordinates. In the next section, we will show that the metric is Kähler such that $g_{rs} = g_{\bar{r}\bar{s}} = 0$. We also have that $g_{r\bar{s}}$ is Hermitian, i.e. $g_{r\bar{s}} = \bar{g}_{s\bar{r}}$, since the metric is real. The metric (3.46) then becomes

$$ds^2 = \sum_{r,s=1}^{\mathcal{N}} g_{r\bar{s}}\, dZ_r d\bar{Z}_s, \tag{3.47}$$

[2]This moduli space approximation is considered good provided the field excitations normal to $\mathcal{M}_{\mathcal{N}}$ are negligible.

and the resulting Lagrangian is

$$L = \frac{1}{2} \sum_{r,s=1}^{\mathcal{N}} g_{r\bar{s}} \dot{Z}_r \dot{\bar{Z}}_s. \tag{3.48}$$

This results in geodesic motion at constant speed on the moduli space $\mathcal{M}_{\mathcal{N}}$ [2].

3.6 Metric on the moduli space $\mathcal{M}_{\mathcal{N}}$

In this section, we follow the work of Samols [23], who determined an expression for the Kähler metric $g_{r\bar{s}}$, and the associated Kähler form ω, on the $\mathcal{N}$-vortex moduli space $\mathcal{M}_{\mathcal{N}}$. Quite remarkably, the metric can be expressed by analyzing *local* data around the $\mathcal{N}$ distinct vortices in the Higgs field ϕ. It will be shown that the metric on $\mathcal{M}_{\mathcal{N}} \backslash \Delta_{\mathcal{N}}$ has a smooth extension to the complete moduli space $\mathcal{M}_{\mathcal{N}}$ [27]. Our aim is to express the kinetic energy (3.45) in terms of $\{\dot{Z}_r\}$ and $\{\dot{\bar{Z}}_r\}$ for $1 \leq r \leq \mathcal{N}$, and thus determine the form of the metric on $\mathcal{M}_{\mathcal{N}}$.

3.6.1 Collective coordinates and fields

Let the time-dependent field configurations $a = \{\phi(t), A_i(t)\}$ be $\mathcal{N}$-vortex solutions of the Bogomolny equations (3.35) and (3.36), where the distinct[3] vortex centres $\{Z_r(t) : 1 \leq r \leq \mathcal{N}\}$ move slowly. Now suppose the generalized velocity $\partial_0 a = \{\partial_0 \phi, \partial_0 A_i\}$ satisfies Gauss' law (3.44) in the $A_0 = 0$ gauge, then the kinetic energy is

$$T[\phi, A] = \frac{1}{2} \int_{\mathbb{R}^2} \left(\partial_0 A_i \partial_0 A_i + \partial_0 \bar{\phi} \partial_0 \phi \right) \, d^2x. \tag{3.49}$$

Using the complex gauge field components (3.33), the kinetic energy can be expressed as

$$T[\phi, A] = \frac{1}{2} \int_{\mathbb{C}} \left(4 \partial_0 A_{\bar{z}} \partial_0 A_z + \partial_0 \bar{\phi} \partial_0 \phi \right) \, d^2x \tag{3.50}$$

As we did in Section 3.3, we can express the Higgs field ϕ in terms of a gauge invariant quantity h via $h = \log |\phi|^2$, so that $\phi = e^{\frac{1}{2}h + i\varphi}$ where φ is a phase factor [16]. Further, define [2]

$$\eta = \partial_0 \left(\log \phi \right) = \frac{1}{2} \partial_0 h + i \partial_0 \varphi, \tag{3.51}$$

such that

$$\partial_0 \phi = \phi \eta. \tag{3.52}$$

Note that η has logarithmic singularities at the zeros of the Higgs field ϕ. Then, as we previously did, the first Bogomolny equation (3.35) can be used to eliminate the gauge field. This yields $A_{\bar{z}} = -i \partial_{\bar{z}} \log \phi$ and, by complex conjugation, $A_z = i \partial_z \log \bar{\phi}$. Taking the time derivative of the gauge field components $(A_z, A_{\bar{z}})$, and using the relation (3.52), yields

$$\partial_0 A_{\bar{z}} = -i \partial_{\bar{z}} \eta, \quad \partial_0 A_z = i \partial_z \bar{\eta}. \tag{3.53}$$

Therefore, the first term in the kinetic energy integrand can be written in terms of η by

$$\partial_0 A_{\bar{z}} \partial_0 A_z = \partial_z \bar{\eta} \partial_{\bar{z}} \eta. \tag{3.54}$$

Similarly, from (3.52), the second term becomes

$$\partial_0 \bar{\phi} \partial_0 \phi = e^h \bar{\eta} \eta. \tag{3.55}$$

[3] Assuming the zeros are distinct means the zeros all have multiplicity one.

Hence, the kinetic energy (3.50) can now be expressed in terms of η as

$$T = \frac{1}{2} \int_{\mathbb{C}} \left(4\partial_z \bar{\eta} \partial_{\bar{z}} \eta + e^h \bar{\eta} \eta \right) \, d^2x. \tag{3.56}$$

Now, away from the zeros of the Higgs field ϕ, we saw that the second Bogomolny equation (3.36) reduces to Taubes's equation in the form

$$\nabla^2 h + 1 - e^h = 0, \tag{3.57}$$

where $\nabla^2 = 4\partial_z \partial_{\bar{z}}$ is the naive Laplacian. The time derivative of Taubes' equation (3.57) gives

$$\left(\nabla^2 - e^h \right) \partial_0 h = 0. \tag{3.58}$$

In terms of the complex gauge field components $(A_z, A_{\bar{z}})$, Gauss' law (3.44) is given by

$$2 \left(\partial_z \partial_0 A_{\bar{z}} + \partial_{\bar{z}} \partial_0 A_z \right) + \frac{i}{2} (\bar{\phi} \partial_0 \phi - \phi \partial_0 \bar{\phi}) = 0. \tag{3.59}$$

Using relations (3.51)–(3.53), this becomes

$$\begin{aligned} 2 \left(\partial_z \partial_0 A_{\bar{z}} + \partial_{\bar{z}} \partial_0 A_z \right) + \frac{i}{2} (\bar{\phi} \partial_0 \phi - \phi \partial_0 \bar{\phi}) &= -2i \partial_z \partial_{\bar{z}} \eta + 2i \partial_{\bar{z}} \partial_z \bar{\eta} + \frac{i}{2} \left(\bar{\phi} \phi \eta - \phi \bar{\phi} \bar{\eta} \right) \\ &= \nabla^2 \partial_0 \varphi - e^h \partial_0 \varphi, \end{aligned} \tag{3.60}$$

and thus Gauss' law reduces to

$$\left(\nabla^2 - e^h \right) \partial_0 \varphi = 0. \tag{3.61}$$

From the definition (3.51), we obtain, upon combining (3.58) with (3.61),

$$\left(\nabla^2 - e^h \right) \eta = 0, \tag{3.62}$$

which is only valid away from the zeros of the Higgs field ϕ. Finite energy solutions impose the boundary condition $h \to 0$ as $|z| \to \infty$. We also require that $|\eta| \to 0$ as $|z| \to \infty$, so that the generalized velocity $\partial_0 a$ is finite in the metric [23]. The operator $-\nabla^2 + e^h$ is a non-singular positive definite operator [2], so any non-trivial solutions must have logarithmic singularities at the vortex centres. We now want to extend the equation for η to include the vortex centres, and we note that from Taubes's equation (3.39) we already have the form of h on $\mathbb{C}$.

In the neighbourhood of a slowly moving zero (with multiplicity one), say Z_r, the Higgs field ϕ can be given by

$$\phi = (z - Z_r) e^k, \tag{3.63}$$

where k is smooth and finite [2]. This implies that

$$\log \phi = \log(z - Z_r) + k, \tag{3.64}$$

and so, in the vicinity of a simple zero Z_r, η takes the form

$$\eta = \frac{-\dot{Z}_r}{z - Z_r} + O(1). \tag{3.65}$$

Therefore, we see that η has simple poles at the vortex centres $\{Z_r\}$ with corresponding residues $\{-\dot{Z}_r\}$ for $1 \leq r \leq \mathcal{N}$. Around the vortex centre Z_r, $e^h \sim |z - Z_r|^2$ and so $e^h \eta$ has no singularity here. However, this is not the same for the $\nabla^2 \eta$ term, as we will see below. Now, in two dimensions, we have the following result [2, 23]

$$\nabla^2 \log |z - Z_r|^2 = 4\pi \delta^2 (z - Z_r). \tag{3.66}$$

Differentiating (3.66) with respect to Z_r yields

$$\frac{\partial}{\partial Z_r}\nabla^2 \log|z-Z_r|^2 = \nabla^2\left(\frac{-1}{z-Z_r}\right) = 4\pi\frac{\partial}{\partial Z_r}\delta^2(z-Z_r) = -4\pi\partial_z\delta^2(z-Z_r). \tag{3.67}$$

Thus, the extension of (3.62) to the whole plane is given by

$$\left(\nabla^2 - e^h\right)\eta = -4\pi\sum_{r=1}^{\mathcal{N}}\dot{Z}_r\partial_z\delta^2(z-Z_r). \tag{3.68}$$

The solution to the Taubes' equation (3.39) can be related to the solution of (3.68) in the following way. Taking the Z_r derivative of Taubes' equation,

$$\nabla^2 h + 1 - e^h = 4\pi\sum_{r=1}^{\mathcal{N}}\delta^2(z-Z_r), \tag{3.69}$$

we obtain

$$\left(\nabla^2 - e^h\right)\frac{\partial h}{\partial Z_r} = -4\pi\partial_z\delta^2(z-Z_r). \tag{3.70}$$

From the linearity of (3.68) [23], we deduce that

$$\eta = \sum_{r=1}^{\mathcal{N}}\dot{Z}_r\frac{\partial h}{\partial Z_r}. \tag{3.71}$$

3.6.2 Kähler form of the metric

Consider the kinetic energy functional (3.56), defined as an integral over the whole plane. Our focus now is to express this in terms of local data in the vicinity of the $\mathcal{N}$ distinct vortex centres. Let's decompose the plane $\mathbb{C}$ into two regions, $\mathbb{C}\backslash D$ and D, and let

$$D = \bigcup_{r=1}^{\mathcal{N}} D_r \tag{3.72}$$

be union of non-overlapping small disks D_r of radius ϵ with boundary C_r, centred at Z_r [23]. The kinetic energy then divides up into two parts:

$$T = \frac{1}{2}\int_{\mathbb{C}\backslash D}\left(4\partial_z\bar{\eta}\partial_{\bar{z}}\eta + e^h\bar{\eta}\eta\right)\,d^2x + \frac{1}{2}\int_D\left(4\partial_z\bar{\eta}\partial_{\bar{z}}\eta + e^h\bar{\eta}\eta\right)\,d^2x. \tag{3.73}$$

The second integral is over the small disks D_r in the neighbourhood of the vortex centres, and so vanishes in the limit $\epsilon \to 0$. Since η is smooth in the region $\mathbb{C}\backslash D$, we can express the first integral as

$$\begin{aligned} T &= \frac{1}{2}\int_{\mathbb{C}\backslash D}\left(4\partial_z\bar{\eta}\partial_{\bar{z}}\eta + e^h\bar{\eta}\eta\right)\,d^2x \\ &= \frac{1}{2}\int_{\mathbb{C}\backslash D}\left(4\partial_z(\bar{\eta}\partial_{\bar{z}}\eta) - 4\bar{\eta}\partial_z\partial_{\bar{z}}\eta + e^h\bar{\eta}\eta\right)\,d^2x \\ &= 2\int_{\mathbb{C}\backslash D}\partial_z(\bar{\eta}\partial_{\bar{z}}\eta)\,d^2x - \frac{1}{2}\int_{\mathbb{C}\backslash D}\bar{\eta}\left(\nabla^2 - e^h\right)\eta\,d^2x. \end{aligned} \tag{3.74}$$

As the integral region $\mathbb{C}\backslash D$ is away from the vortex centres, η thus satisfies (3.62) and so the kinetic energy reduces to

$$T = 2\int_{\mathbb{C}\backslash D}\partial_z(\bar{\eta}\partial_{\bar{z}}\eta)\,d^2x. \tag{3.75}$$

Now, consider a domain S in $\mathbb{C}$ with boundary ∂S traced anticlockwise, and let $f(z, \bar{z})$ be a differentiable function. Then, using the identity [2]

$$\int_S \partial_i f \, d^2x = \oint_{\partial S} f n_i \, dl, \tag{3.76}$$

we have the following result

$$\int_S \partial_z f \, d^2x = \frac{1}{2}\int_S (\partial_1 f - i\partial_2 f) \, d^2x = \frac{1}{2}\oint_{\partial S} (n_1 - in_2) f \, dl = \frac{i}{2}\oint_{\partial S} f \, d\bar{z}. \tag{3.77}$$

Using the identity (3.77), the integral over $\mathbb{C}\backslash D$ can be transformed into a contour integral around the circle at infinity S^1_∞ plus the contour integrals around the disk boundaries C_r (traced anticlockwise). This is illustrated in Fig. 4. The kinetic energy (3.75) splits as follows

$$\begin{aligned} T &= i\sum_{r=1}^{\mathcal{N}} \oint_{C_r} \bar{\eta}\partial_{\bar{z}}\eta \, d\bar{z} + i\oint_{S^1_\infty} \bar{\eta}\partial_{\bar{z}}\eta \, d\bar{z} \\ &= -i\sum_{r=1}^{\mathcal{N}} \oint_{C_r} \bar{\eta}\partial_{\bar{z}}\eta \, d\bar{z}, \end{aligned} \tag{3.78}$$

where the contribution from S^1_∞ vanishes by virtue of the boundary condition $|\eta| \to 0$ as $|z| \to \infty$. Note that the minus sign in (3.78) accounts for taking the contour integral anticlockwise around the disk boundary C_r.

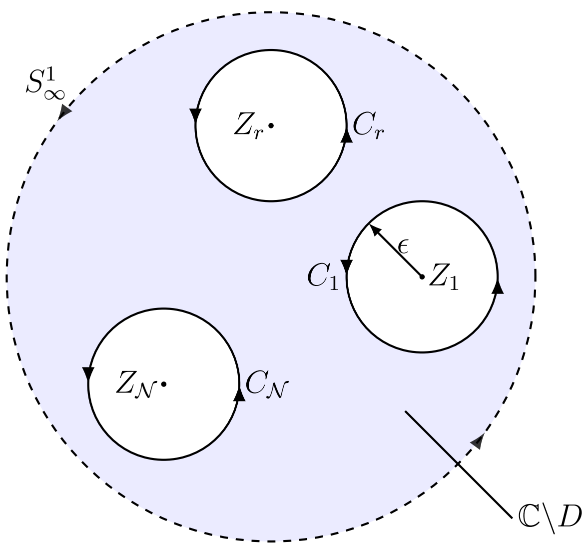


Figure 4: The integration region $\mathbb{C}\backslash D$, where D is the union of non-overlapping disks D_r of radius ϵ with boundary C_r traced anticlockwise, centred on Z_r.

We will now follow the Green's function approach set out by Manton [2] to show that the kinetic energy T is real. Around the vortex centre Z_r, in the region enclosed by $|z - Z_r| = \epsilon$, we can expand $h(z, \bar{z})$ as

$$h = \log|z-Z_r|^2 + a_r + \frac{1}{2}\bar{b}_r(z-Z_r) + \frac{1}{2}b_r(\bar{z}-\bar{Z}_r) + \bar{c}_r(z-Z_r)^2 + d_r|z-Z_r|^2 + c_r(\bar{z}-\bar{Z}_r)^2 + O(\epsilon^3), \tag{3.79}$$

where $a_r, d_r \in \mathbb{R}$ and $b_r, c_r \in \mathbb{C}$ [3]. The logarithmic term and the coefficient d_r are determined locally. In fact, in order for the expansion (3.79) to satisfy Taubes' equation (3.69), we require $d_r = -\frac{1}{4}$ [23]. The remaining coefficients are all determined by the positions of the other vortices Z_s

$(s \neq r)$, but not in an explicitly known way [22]. In particular, b_r (and $\bar{b}_r$) is the key coefficient – it plays a central rôle in the form of the metric. Geometrically, b_r measures the deviation of contours of h around Z_r from circles centred at Z_r due to interactions with other vortices [16]. We will show that certain symmetry properties of the derivatives of b_r imply that the kinetic energy T is real and, most importantly, that the metric is Kähler.

The Green's function construction is as follows. Recall equation (3.70), we will show that $\frac{\partial h}{\partial Z_r}$ behaves like a Green's function for the operator $\nabla^2 - e^h$. Consider the integral, for $r \neq s$ [2],

$$\begin{aligned} \int_{\mathbb{C}} \frac{\partial h}{\partial Z_s}(\nabla^2 - e^h)\frac{\partial h}{\partial Z_r}\, d^2x &= -4\pi \int_{\mathbb{C}} \frac{\partial h}{\partial Z_s}\partial_z \delta^2(z - Z_r)\, d^2x \\ &= 4\pi \int_{\mathbb{C}} \partial_z \frac{\partial h}{\partial Z_s}\delta^2(z - Z_r)\, d^2x \\ &= 4\pi\, \partial_z \frac{\partial h}{\partial Z_s}\bigg|_{z=Z_r}, \end{aligned} \tag{3.80}$$

where we have used integration by parts so that ∂_z operates on $\frac{\partial h}{\partial Z_s}$. Thus, we see that $\frac{\partial h}{\partial Z_r}$ does indeed manifest a type of Green's function for $\nabla^2 - e^h$. Similarly, we can use integration by parts so that $\nabla^2 - e^h$ acts on $\frac{\partial h}{\partial Z_s}$ instead. It is easy to check that this results in the same form (3.80), but with the rôle of Z_r and Z_s swapped. This yields the symmetry property

$$\partial_z \frac{\partial h}{\partial Z_s}\bigg|_{z=Z_r} = \partial_z \frac{\partial h}{\partial Z_r}\bigg|_{z=Z_s}. \tag{3.81}$$

In a similar manor, we can apply the exact same Green's function approach to the complex conjugate of (3.70). This yields another symmetry property,

$$\partial_z \frac{\partial h}{\partial \bar{Z}_s}\bigg|_{z=Z_r} = \partial_{\bar{z}} \frac{\partial h}{\partial Z_r}\bigg|_{z=Z_s}, \tag{3.82}$$

which is also valid for $r = s$ [2].

It should be clear by now that, from the expansion (3.79), the symmetry relations derived above will lead us to the desired symmetry relations for the derivatives of b_r and b_s. Writing the symmetry relation (3.82) out explicitly, and using the expansion (3.79), gives us

$$\partial_{\bar{z}} \frac{\partial h}{\partial Z_r}\bigg|_{z=Z_s} = \partial_{\bar{z}} \left(\frac{\partial a_s}{\partial Z_r} + \frac{1}{2}\frac{\partial \bar{b}_s}{\partial Z_r}(z - Z_s) + \frac{1}{2}\frac{\partial b_s}{\partial Z_r}(\bar{z} - \bar{Z}_s) + O(|z - Z_s|^2) \right)\bigg|_{z=Z_s} = \frac{1}{2}\frac{\partial b_s}{\partial Z_r} \tag{3.83}$$

and

$$\partial_z \frac{\partial h}{\partial \bar{Z}_s}\bigg|_{z=Z_r} = \partial_z \left(\frac{\partial a_r}{\partial \bar{Z}_s} + \frac{1}{2}\frac{\partial \bar{b}_r}{\partial \bar{Z}_s}(z - Z_r) + \frac{1}{2}\frac{\partial b_r}{\partial \bar{Z}_s}(\bar{z} - \bar{Z}_r) + O(|z - Z_r|^2) \right)\bigg|_{z=Z_r} = \frac{1}{2}\frac{\partial \bar{b}_r}{\partial \bar{Z}_s}. \tag{3.84}$$

From (3.82), the key symmetry relation follows, viz.

$$\frac{\partial \bar{b}_r}{\partial \bar{Z}_s} = \frac{\partial b_s}{\partial Z_r} \tag{3.85}$$

If we apply the same procedure to the first symmetry relation (3.81), we get the resulting symmetry relation

$$\frac{\partial \bar{b}_r}{\partial Z_s} = \frac{\partial \bar{b}_s}{\partial Z_r}. \tag{3.86}$$

Both symmetry relations (3.85) and (3.86) are valid for all r and s [2]. The importance of the symmetry relation (3.85) becomes manifest when we arrive at the form of the metric. As we will show below, the symmetry relation (3.85) implies that the kinetic energy T is real and, consequently, the metric is Kähler.

Returning from our brief digression, we now want to evaluate (3.78) in terms of the symmetry relation (3.85). In the neighbourhood of a vortex centre Z_s, say the region bounded by the circle $|z - Z_s| = \epsilon$, the Taylor expansion of h is given by [23]

$$h = \log|z-Z_s|^2 + a_s + \frac{1}{2}\bar{b}_s(z-Z_s) + \frac{1}{2}b_s(\bar{z}-\bar{Z}_s) + \bar{c}_s(z-Z_s)^2 - \frac{1}{4}|z-Z_s|^2 + c_s(\bar{z}-\bar{Z}_s)^2 + O(\epsilon^3). \quad (3.87)$$

In order to explicitly calculate η via (3.71), we need to determine $\frac{\partial h}{\partial Z_r}$. Differentiating the expansion (3.87) with respect to vortex centre Z_r yields

$$\frac{\partial h}{\partial Z_r} = -\frac{\delta_{rs}}{z-Z_s} + \frac{\partial a_s}{\partial Z_r} + \frac{1}{2}\frac{\partial \bar{b}_s}{\partial Z_r}(z-Z_s) + \frac{1}{2}\frac{\partial b_s}{\partial Z_r}(\bar{z}-\bar{Z}_s) - \frac{1}{2}\bar{b}_s\delta_{rs} + \frac{1}{4}(\bar{z}-\bar{Z}_s)\delta_{rs}$$
$$- 2\bar{c}_s(z-Z_s)\delta_{rs} + O(\epsilon^2). \quad (3.88)$$

This gives the explicit form for η [2], upon substitution into (3.71),

$$\eta = \sum_{r=1}^{\mathcal{N}} \dot{Z}_r \frac{\partial h}{\partial Z_r} = \sum_{r=1}^{\mathcal{N}} \dot{Z}_r\left(-\frac{\delta_{rs}}{z-Z_s}\right) + O(1) = -\frac{\dot{Z}_s}{z-Z_s} + O(1). \quad (3.89)$$

The terms in the kinetic energy integral are $\bar{\eta}$ and $\partial_{\bar{z}}\eta$. From (3.89), $\bar{\eta}$ is simply the complex conjugate of this, i.e.

$$\bar{\eta} = -\frac{\dot{\bar{Z}}_s}{\bar{z}-\bar{Z}_s} + O(1). \quad (3.90)$$

In the vicinity of Z_s, from (3.88), we have

$$\partial_{\bar{z}}\frac{\partial h}{\partial Z_r} = \frac{1}{2}\frac{\partial b_s}{\partial Z_r} + \frac{1}{4}\delta_{rs} + O(\epsilon). \quad (3.91)$$

Hence,

$$\partial_{\bar{z}}\eta = \sum_{r=1}^{\mathcal{N}} \dot{Z}_r \partial_{\bar{z}}\frac{\partial h}{\partial Z_r} = \sum_{r=1}^{\mathcal{N}} \dot{Z}_r\left(\frac{1}{2}\frac{\partial b_s}{\partial Z_r} + \frac{1}{4}\delta_{rs}\right) + O(\epsilon). \quad (3.92)$$

Therefore, the term in the kinetic energy integral (3.78) becomes

$$\bar{\eta}\partial_{\bar{z}}\eta = -\sum_{r=1}^{\mathcal{N}} \dot{\bar{Z}}_s\dot{Z}_r\left(\frac{1}{2}\frac{\partial b_s}{\partial Z_r} + \frac{1}{4}\delta_{rs}\right)\frac{1}{\bar{z}-\bar{Z}_s} + O(\epsilon) \quad (3.93)$$

In the above form, we see that this expression has a simple pole with associated residue given by the coefficient attached to $(\bar{z}-\bar{Z}_s)^{-1}$. So, using the residue theorem (for an anti-holomorphic function), the contour integral around C_s becomes

$$\oint_{C_s} \bar{\eta}\partial_{\bar{z}}\eta\, d\bar{z} = -2\pi i\left[-\dot{\bar{Z}}_s\sum_{r=1}^{\mathcal{N}} \dot{Z}_r\left(\frac{1}{2}\frac{\partial b_s}{\partial Z_r} + \frac{1}{4}\delta_{rs}\right)\right]$$
$$= 2\pi i\dot{\bar{Z}}_s\sum_{r=1}^{\mathcal{N}} \dot{Z}_r\left(\frac{1}{2}\frac{\partial b_s}{\partial Z_r} + \frac{1}{4}\delta_{rs}\right). \quad (3.94)$$

Finally, with the contribution from all circles C_s, the kinetic energy (3.78) reduces to

$$T = \frac{1}{2}\pi\sum_{r,s=1}^{\mathcal{N}}\left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right)\dot{Z}_r\dot{\bar{Z}}_s. \quad (3.95)$$

This is the fundamental result that we have been striving towards. We have shown that although the metric was initially defined defined as an integral over the plane, it can remarkably be expressed in terms of local data in the vicinity of each vortex. By the symmetry property (3.85), it is clear that the kinetic energy T has to be real. It immediately follows that the metric on the $\mathcal{N}$-vortex moduli space $\mathcal{M}_{\mathcal{N}}$,

$$ds^2 = \sum_{r,s=1}^{\mathcal{N}} g_{r\bar{s}}\, dZ_r d\bar{Z}_s = \pi \sum_{r,s=1}^{\mathcal{N}} \left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r} \right) dZ_r d\bar{Z}_s \tag{3.96}$$

is not just real, but it is also Kähler [16, 22] (this is shown below).

If we consider well separated vortices, then the contribution from b_s and its derivatives are negligible. In this case, the metric simply reduces to the flat metric on $\mathbb{C}^{\mathcal{N}}$ times the vortex mass π, i.e.

$$ds^2 = \pi \sum_{r=1}^{\mathcal{N}} dZ_r d\bar{Z}_r. \tag{3.97}$$

This describes the metric on the asymptotic form of the moduli space $\mathcal{M}_{\mathcal{N}} \cong \mathbb{C}^{\mathcal{N}}/S_{\mathcal{N}}$, since the vortices are indistinguishable [2].

The Kähler 2-form associated with the metric (3.96) is [27]

$$\omega = \frac{i\pi}{2} \sum_{r,s=1}^{\mathcal{N}} \left(\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r} \right) dZ_r \wedge d\bar{Z}_s. \tag{3.98}$$

For a metric to be Kähler, the Kähler form (3.98) must be closed, i.e. $d\omega = 0$ [2]. So, taking the exterior derivative of the 2-form (3.98) gives [23]

$$d\omega = i\pi \sum_{r,s,t=1}^{\mathcal{N}} \left[\frac{\partial^2 \bar{b}_r}{\partial \bar{Z}_t \partial \bar{Z}_s} d\bar{Z}_t \wedge dZ_r \wedge d\bar{Z}_s + \frac{\partial^2 b_s}{\partial Z_t \partial Z_r} dZ_t \wedge dZ_r \wedge d\bar{Z}_s \right] = 0, \tag{3.99}$$

by virtue of the symmetry property (3.85), and hence the metric is indeed Kähler.

3.6.3 Kähler potential $\mathcal{K}$ for vortices on $\mathbb{C}$

One of the main consequences of the hermicity identity (3.85) and the symmetry identity (3.86) is that there exists a locally real function $\tilde{\mathcal{K}}$ of the collective coordinates, such that [16]

$$\frac{\partial \tilde{\mathcal{K}}}{\partial \bar{Z}_r} = b_r, \quad \frac{\partial \tilde{\mathcal{K}}}{\partial Z_r} = \bar{b}_r. \tag{3.100}$$

In terms of the function $\tilde{\mathcal{K}}$, the metric is

$$ds^2 = \pi \sum_{r,s=1}^{\mathcal{N}} \left(\delta_{rs} + 2\frac{\partial^2 \tilde{\mathcal{K}}}{\partial Z_r \partial \bar{Z}_s} \right) dZ_r d\bar{Z}_s. \tag{3.101}$$

Chen and Manton [16] evaluated the Kähler potential, using a scaling argument, and showed that the Kähler potential for vortices restricted to the scaling motion is given by

$$\mathcal{K} = \pi \sum_{r=1}^{\mathcal{N}} Z_r \bar{Z}_r + 2\pi \tilde{\mathcal{K}} \tag{3.102}$$

$$= \pi \sum_{r=1}^{\mathcal{N}} Z_r \bar{Z}_r - 2\pi \int_{\infty}^{1} \frac{d\tau}{\tau} \left\{ \frac{1}{\pi} \int h(\mathbf{x};\tau)\, d^2x + 6\mathcal{N} \right\}, \tag{3.103}$$

where the dimensionless parameter τ measures the deviation of $\tilde{\mathcal{K}}$ under an overall scaling of the vortex moduli space.

3.6.4 Centre of mass motion

By considering a translation of all the vortices and noting that $b_r \to 0$ as the vortices separate, the translational invariance of the entire system implies that [16]

$$\sum_{r}^{\mathcal{N}} b_r = \sum_{r}^{\mathcal{N}} \bar{b}_r = 0. \tag{3.104}$$

Similarly, the rotational invariance of the system implies the following result [28]

$$\sum_{r=1}^{\mathcal{N}} \left(b_r Z_r - \bar{b}_r \bar{Z}_r \right) = 0, \tag{3.105}$$

and so the quantity $\sum_{r=1}^{\mathcal{N}} b_r Z_r$ is real. This enables us to define a centre of mass coordinate $Z = \frac{1}{\mathcal{N}}(\sum_r Z_r)$ and a set of relative coordinates $\zeta_r = Z_r - Z, 1 \leq r \leq \mathcal{N} - 1$ which are unaffected by an overall translation [2]. The metric then separates in the form

$$ds^2 = \mathcal{N}\pi dZ d\bar{Z} + \sum_{r,s=1}^{\mathcal{N}-1} \tilde{g}_{r\bar{s}}\, d\zeta_r d\bar{\zeta}_s \tag{3.106}$$

and, thus, $\mathcal{M}_{\mathcal{N}}$ decomposes as an isometric product

$$\mathcal{M}_{\mathcal{N}} = \mathbb{C} \times \mathcal{M}^0_{\mathcal{N}}, \tag{3.107}$$

where $\mathcal{M}^0_{\mathcal{N}}$ is the space of $\mathcal{N}$-vortices with fixed centre [23].

3.6.5 2-Vortex metric on $\mathcal{M}_2$

The moduli space $\mathcal{M}_2$ decomposes into two 2-dimensional spaces $\mathcal{M}_2 = \mathbb{C} \times \mathcal{M}^0_2$. Samols [23] showed the decoupling of the centre of mass motion, and calculated that the metric (3.106) reduces to

$$ds^2 = 2\pi dZ d\bar{Z} + F^2(|\zeta|) d\zeta d\bar{\zeta}. \tag{3.108}$$

It is convenient to introduce the polar coordinates (ρ, θ) via $\zeta = \rho e^{i\theta}$, so that the vortices are located at the positions $\zeta_1 = Z + \rho e^{i\theta}$ and $\zeta_2 = Z - \rho e^{i\theta}$. Considering the relative motion of the vortices, the metric becomes [2]

$$ds^2_{\text{rel}} = F^2(\rho) \left(d\rho^2 + \rho^2 d\theta^2 \right), \tag{3.109}$$

where [22]

$$F^2(\rho) = 2\pi \left(1 + \frac{1}{\rho} \frac{d}{d\rho} (\rho b(\rho)) \right). \tag{3.110}$$

Here, the coefficients b_1 and b_2 occur in the expansion of h about the vortex centres, and the rotation and parity symmetries imply that $b_1 = -b_2 = b(\rho) e^{i\theta}$, where $b(\rho) \in \mathbb{R}$. Furthermore, Samols [23] numerically determined the variable $b(\rho)$ and thus $F^2(\rho)$.

We can represent the metric (3.109) by isometrically embedding the manifold $\mathcal{M}^0_2$ in $\mathbb{R}^3$ as a surface of revolution [23]. The manifold $\mathcal{M}^0_2$ is a smoothed cone of half-opening angle 30° [2]. The geodesics passing over the top of the smoothed cone describe the right-angle scattering of two vortices, as can be seen in Fig. 5. It can be shown (see e.g. [2, 22, 23]) that this surface is asymptotic to the singluar cone C_2. The geodesics on the cone C_2 describe the non-interaction of a pair of indistinguishable point-particles, therefore illustrating our earlier statement in regards to the 'point-particle' metric and the vortex metric agreeing asymptotically.

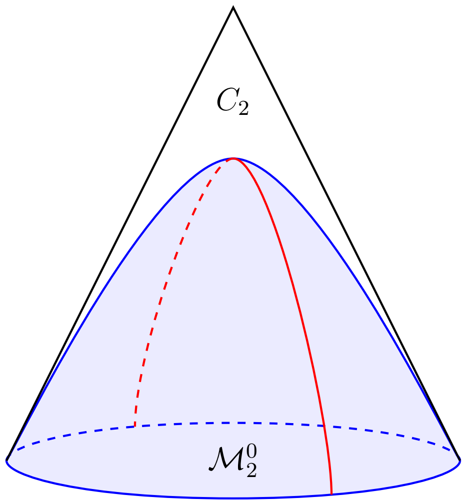


Figure 5: Representation of the metric ds^2_{rel} by isometrically embedding the manifold $\mathcal{M}_2^0$ as a surface of revolution in $\mathbb{R}^3$.

4 Abelian Higgs vortices on $\mathbb{H}^2$

So far, we have considered the interactions of critically coupled vortices in the plane $\mathbb{R}^2$. However, we are now going to consider the further generalization in which we discuss critically coupled vortices on a general, compact Riemann surface M, without boundary. In particular, we are going to consider the mathematically interesting case of Abelian Higgs vortices on the hyperbolic plane $\mathbb{H}^2$ of curvature $-\frac{1}{2}$. It will be shown that, at critical coupling, the Bogomolny equations for vortices on $\mathbb{H}^2$ reduce to the integrable Liouville's equation, where explicit vortex solutions are readily obtainable.

4.1 Hyperbolic vortices

Let M be a compact 2-dimensional Riemann surface, or an open Riemann surface with a boundary at infinity, equipped with a metric compatible with its complex structure [32]. In isothermal coordinates[4] (x^1, x^2) [2], let M have a compatible Riemannian metric with position-dependent conformal factor Ω_M,

$$ds_0^2 = \Omega_M(x^1, x^2)\left((dx^1)^2 + (dx^2)^2\right) = \Omega_M(z, \bar{z})\, dz\, d\bar{z}, \tag{4.1}$$

where $z = x^1 + ix^2$ is a local complex coordinate. Then spacetime is $\mathbb{R} \times M$ with the metric

$$ds^2 = dt^2 - \Omega_M(x^1, x^2)\left((dx^1)^2 + (dx^2)^2\right). \tag{4.2}$$

The fields are locally a $U(1)$ gauge potential A and a complex Higgs field ϕ. Globally, A is a connection 1-form $A = A_1\, dx^1 + A_2\, dx^2$ on a $U(1)$ line bundle $\mathcal{B}$ over M, with fibre $\mathbb{C}$, and ϕ is a section of the bundle $\mathcal{B}$ [18, 32]. The connection has 2-form field strength (curvature of the connection) $F = dA = (\partial_1 A_2 - \partial_2 A_1)\, dx^1 \wedge dx^2$. The first Chern number of the bundle is

$$\mathcal{N} = \frac{1}{2\pi}\int_M F = \frac{1}{2\pi}\int_M F_{12}\, d^2x, \tag{4.3}$$

which takes a positive integer value, and is the only topological invariant of the bundle $\mathcal{B}$ [2]. This is also true for non-compact Riemann surfaces provided the Higgs fields satisfy appropriate boundary conditions [18]. As before, the vortex centres are points on the surface M where the Higgs field ϕ vanishes, and the first Chern number of the bundle $\mathcal{N}$ is the vortex number (counted with multiplicity). Moreover, the multiplicites are all positive which is ensured by the first Bogomolny equation.

[4]Locally, on a Riemannian manifold, coordinates which are isothermal means that the metric is conformal to the Euclidean metric.

Recall the theory of scalar electrodynamics on curved spacetimes (refer to Section 2.2.1). Then we have a Riemannian manifold M equipped with the Riemannian metric $h_{ij}(x^1, x^2) = \Omega_M \, \delta_{ij}$. The natural measure on M is then $\Omega_M \, d^2x$ and the Lagrangian for scalar electrodynamics on M is given by

$$L[\phi, A] = \int_M \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \overline{D_\mu \phi} D^\mu \phi - \frac{\lambda}{8} (1 - |\phi|^2)^2 \right) \Omega_M \, d^2x. \tag{4.4}$$

We can split this Lagrangian in terms of its kinetic and potential energies, $L = T - V$. The corresponding kinetic energy functional is given by

$$T[\phi, A] = \frac{1}{2} \int_M \left(E_1^2 + E_2^2 + \Omega_M \overline{D_0 \phi} D_0 \phi \right) \, d^2x \tag{4.5}$$

and the potential energy functional

$$V[\phi, A] = \frac{1}{2} \int_M \left(\frac{F_{12}^2}{\Omega_M} + \overline{D_1 \phi} D_1 + \overline{D_2 \phi} D_2 + \frac{\lambda \Omega_M}{4} (1 - |\phi|^2)^2 \right) \, d^2x. \tag{4.6}$$

Working in the $A_0 = 0$ gauge, then the static energy functional is $E = V$. Applying the Bogomolny argument to E at critical coupling, $\lambda = 1$, we obtain

$$\begin{aligned} E[\phi, A] = \frac{1}{2} \int_M &\left\{ \frac{1}{\Omega_M} \left[F_{12} - \frac{\Omega_M}{2} \left(1 - |\phi|^2\right) \right]^2 + \left(\overline{D_1 \phi} - i \overline{D_2 \phi} \right) \left(D_1 \phi + i D_2 \phi \right) \right\} \, d^2x \\ &+ \frac{1}{2} \int_M \left\{ i \left(\overline{D_2 \phi} D_1 \phi - \overline{D_2 \phi} D_1 \phi \right) + F_{12} \left(1 - |\phi|^2 \right) \right\} \, d^2x. \end{aligned} \tag{4.7}$$

Then, as we did in the flat plane $\mathbb{R}^2$ case in Section 3.3, we can use the commutation relation (2.41) and the identity (3.24) to rewrite the static energy as

$$\begin{aligned} E[\phi, A] = \pi \mathcal{N} + \frac{1}{2} \int_M &\left\{ \frac{1}{\Omega_M} \left[F_{12} - \frac{\Omega_M}{2} \left(1 - |\phi|^2\right) \right]^2 + \left(\overline{D_1 \phi} - i \overline{D_2 \phi} \right) \left(D_1 \phi + i D_2 \phi \right) \right\} \, d^2x \\ &- \frac{i}{2} \int_M \left\{ \partial_1 \left(\bar{\phi} D_2 \phi \right) - \partial_2 \left(\bar{\phi} D_1 \phi \right) \right\} \, d^2x. \end{aligned} \tag{4.8}$$

The last intergral is that of the exact 2-form $d(\bar{\phi} D \phi)$, which vanishes as our compact surface M has no boundary. Therefore, the static energy functional E can be written as

$$E[\phi, A] = \pi \mathcal{N} + \frac{1}{2} \int_M \left\{ \frac{1}{\Omega_M} \left[F_{12} - \frac{\Omega_M}{2} \left(1 - |\phi|^2\right) \right]^2 + \left(\overline{D_1 \phi} - i \overline{D_2 \phi} \right) \left(D_1 \phi + i D_2 \phi \right) \right\} \, d^2x. \tag{4.9}$$

The standard Bogomolny argument used above shows that the static energy E satisfies the usual Bogomolny bound

$$E \geq \pi \mathcal{N}, \tag{4.10}$$

with E being stationary if and only if the Higgs fields satisfy the Bogomolny equations

$$D_1 \phi + i D_2 \phi = 0, \tag{4.11}$$

$$F_{12} - \frac{\Omega_M}{2} (1 - |\phi|^2) = 0. \tag{4.12}$$

Fields which satisfy these Bogomolny equations minimize the static energy E in their homotopy class [24], which is determined by the first Chern number $\mathcal{N}$. Note that, for a general compact Riemann surface M of area $\mathcal{A}_M$, $\mathcal{N}$-vortex solutions only exist if the following bound is satisfied [27]:

$$\mathcal{A}_M \geq 4\pi \mathcal{N}. \tag{4.13}$$

This bound, first calculated by Bradlow [26], can be obtained by integrating the second Bogomolny equation (4.12) over the surface M, which yields

$$2\int_M F_{12}\, d^2x + \int_M |\phi|^2\Omega_M\, d^2x = \int_M \Omega_M\, d^2x. \tag{4.14}$$

The area element is $\Omega_M\, d^2x$ and the first integral term is 2π times the Chern number $\mathcal{N}$, so

$$4\pi\mathcal{N} + \int_M |\phi|^2\Omega_M\, d^2x = \mathcal{A}_M. \tag{4.15}$$

Since the $|\phi|^2\Omega_M$ term is non-negative, we arrive at the Bradlow bound (4.13).

If the Bradlow bound is saturated exactly, i.e. $\mathcal{A}_M = 4\pi\mathcal{N}$, then solutions exist and we are able to solve both Bogomolny equations. In this case, from Eq. (4.15), the Higgs field ϕ vanishes everywhere on M and the first Bogomolny equation (4.11) is solved trivially. The second Bogomolny equation (4.12) reduces to $F_{12} = \frac{\Omega_M}{2}$, so that the magnetic flux density has constant value $\frac{1}{2}$. Since the Higgs field ϕ vanishes everywhere on M, the solutions are merely limiting cases of vortices [18].

As we did previously, we can express the Higgs field ϕ in terms of a single gauge invariant quantity h and a phase factor χ by setting $\phi = e^{\frac{1}{2}h+i\chi}$. Then, from the first Bogomolny equation (4.11), this enables us to express the gauge potential components (A_1, A_2) as

$$A_1 = \frac{1}{2}\partial_2 h + \partial_1\chi, \quad A_2 = -\frac{1}{2}\partial_1 h + \partial_2\chi. \tag{4.16}$$

The second Bogomolny equation (4.12) then becomes the gauge invariant Taubes equation

$$\nabla^2 h + \Omega_M(x^1, x^2)\left(1 - e^h\right) = 4\pi\sum_{r=1}^{\mathcal{N}} \delta^2(\mathbf{x} - \mathbf{X}_r), \tag{4.17}$$

where $\mathbf{X}_r$ are the vortex centres, where the Higgs field ϕ is zero.

It is convenient to introduce a local complex coordinate $z = x^1 + ix^2$ on M and let the vortex positions on the surface M be given by $\{Z_1, \ldots, Z_{\mathcal{N}}\}$. Then we can rewrite the Taubes equation (4.17) in terms of $z, \bar{z}$ as

$$4\partial_z\partial_{\bar{z}} h + \Omega_M(z,\bar{z})\left(1 - e^h\right) = 4\pi\sum_{r=1}^{\mathcal{N}} \delta^2(z - Z_r). \tag{4.18}$$

There is a unique solution to (4.18), provided the strict Bradlow bound, $\mathcal{A} > 4\pi\mathcal{N}$, is satisfied. Expanding the gauge invariant quantity h around the simple zero Z_r of the field ϕ we obtain [16]

$$\begin{aligned} h(z,\bar{z}) = \log|z - Z_r|^2 + a_r + \frac{1}{2}\bar{b}_r(z - Z_r) + \frac{1}{2}b_r(\bar{z} - \bar{Z}_r) + \bar{c}_r(z - Z_r)^2 \\ - \frac{1}{4}\Omega_M(Z_r, \bar{Z}_r)|z - Z_r|^2 + c_r(\bar{z} - \bar{Z}_r)^2 + \cdots . \end{aligned} \tag{4.19}$$

The only difference between h here and the flat space case is that the coefficient $d_r = -\frac{1}{4}\Omega_M(Z_r, \bar{Z}_r)$. Since the vortices are classically indistinguishable, the $\mathcal{N}$-vortex moduli space $\mathcal{M}_{\mathcal{N}}$ is diffeomorphic to the symmetric product $\mathcal{M}^{\mathcal{N}}/S_{\mathcal{N}}$ [27], where, as before, $S_{\mathcal{N}}$ is the permutation group on $\mathcal{N}$ objects. It should be noted that, as in the planar case, the moduli space $\mathcal{M}_{\mathcal{N}}$ has a smooth manifold structure [2]. Samols [23] calculated that the generalized metric on the $\mathcal{N}$-vortex moduli space $\mathcal{M}_{\mathcal{N}}$ is

$$ds^2 = \pi\sum_{r,s=1}^{\mathcal{N}} \left(\Omega_M(Z_r, \bar{Z}_r)\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r}\right) dZ_r d\bar{Z}_s, \tag{4.20}$$

which is again Kähler.

4.1.1 Volume of moduli space $\mathcal{M}_{\mathcal{N}}$

There exists a particularly interesting result of Manton and Nasir [27] in relation to vortices on a *compact* Riemann surface M, of genus g. The statistical mechanics of these vortices [29] essentially reduces down to computing the volume of the moduli space $\mathcal{M}_{\mathcal{N}}$. Remarkably, the volume of the moduli space $\mathcal{M}_{\mathcal{N}}$ can be determined without explicitly knowing the metric. We have already shown that the moduli space $\mathcal{M}_{\mathcal{N}}$ is Kähler, all we need in order to calculate the volume is the associated Kähler 2-form of the Kähler metric (4.20). The associated Kähler form is given by

$$\omega = \frac{i\pi}{2} \sum_{r,s=1}^{\mathcal{N}} \left(\Omega_M(Z_r, \bar{Z}_r)\delta_{rs} + 2\frac{\partial b_s}{\partial Z_r} \right) dZ_r \wedge d\bar{Z}_s. \tag{4.21}$$

By taking the exterior derivative, exaclty as we did in the flat plane case (see calculaton (3.99)), one can show that ω is closed. The volume of the moduli space $\mathcal{M}_{\mathcal{N}}$ is given by [2]

$$\mathrm{Vol}(\mathcal{M}_{\mathcal{N}}) = \int_{\mathcal{M}_{\mathcal{N}}} \frac{\omega^{\mathcal{N}}}{\mathcal{N}!}. \tag{4.22}$$

In the paper by Manton and Nasir [27], they investigate the problem from a cohomological perspective and derive an expression for the cohomological class of the Kähler 2-form (4.21). The key result is the calculation of the following expression for the volume of the moduli space $\mathcal{M}_{\mathcal{N}}$:

$$\mathrm{Vol}(\mathcal{M}_{\mathcal{N}}) = \int_{\mathcal{M}_{\mathcal{N}}} \frac{\omega^{\mathcal{N}}}{\mathcal{N}!} = \pi^{\mathcal{N}} \sum_{j=0}^{g} \frac{(4\pi)^j (\mathcal{A}_M - 4\pi\mathcal{N})^{\mathcal{N}-j} g!}{j(\mathcal{N}-j)!(g-j)!j!}, \tag{4.23}$$

which is only a function of the area of the Riemann surface M, the vortex number $\mathcal{N}$, and its genus g – it does not require any information about the shape of M.

4.2 Vortices on the hyperbolic plane $\mathbb{H}^2$

Now consider the case of Abelian Higgs vortices slowly-moving on the hyperbolic plane $\mathbb{H}^2$. We will show that the coupled Bogomolny equations (4.11)-(4.12) are integrable on the hyperbolic plane $\mathbb{H}^2$ for the particular Gaussian curvature $K = -\frac{1}{2}$. To do this, we reduce the Bogomolny equations for static $\mathcal{N}$-vortices to Liouville's equation on a disk [24], which can be solved exactly. We find that, by introducing a holomorphic mapping $f : \mathbb{H}^2 \to \mathbb{H}^2$, we can construct $\mathcal{N}$-vortex solutions on $\mathbb{H}^2$ in terms of the holomorphic map f, which must be a Blaschke rational function. In this case, the vortex centres are ramification points of the map f, i.e. locations where the derivative of f vanishes.

We begin by using the complex forms of the gauge field (3.33) and the curvature tensor (3.34) to express the Bogomolny equations (4.11) and (4.12) in complex form, viz.

$$D_{\bar{z}}\phi = \partial_{\bar{z}}\phi - iA_{\bar{z}}\phi = 0, \tag{4.24}$$

$$F_{z\bar{z}} = \frac{i\Omega_M}{4}(1 - |\phi|^2). \tag{4.25}$$

It is worth noting that these Bogomolny equations are covariant under conformal transformations.

Now, following Manton and Rink [18], let the Higgs field be $\phi = \sqrt{H}\psi$, where $H(z, \bar{z})$ is a metric on the line bundle $\mathcal{B}$, which is locally a positive real function on M [18], and $\psi(z)$ is a holomorphic function, which we will show below. So far, we have been working in the unitary gauge with $H = 1$. However, we now have a larger gauge freedom in the Bogomolny equations and can apply a gauge transformation $g(z, \bar{z}) \in \mathbb{C}^*$, where $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, that has the following effect [18]:

$$\psi \to g\psi, \quad \bar{\psi} \to \bar{g}\bar{\psi}, \tag{4.26}$$

$$A_z \to A_z - i(\partial_z g)g^{-1}, \quad A_{\bar{z}} \to A_{\bar{z}} - i(\partial_{\bar{z}} g)g^{-1}, \tag{4.27}$$

$$H \to g^{-1}\bar{g}^{-1}H. \tag{4.28}$$

We see that the magnitude of the Higgs field $|\phi|^2$,

$$\begin{aligned}|\phi|^2 &\to g^{-1}\bar{g}^{-1}Hg\psi\bar{g}\bar{\psi}\\ &= H\psi\bar{\psi}\\ &= |\phi|^2,\end{aligned} \tag{4.29}$$

and the magnetic field strength $F_{z\bar{z}}$,

$$\begin{aligned}F_{z\bar{z}} &\to \partial_z\left(A_{\bar{z}} - i(\partial_{\bar{z}}g)g^{-1}\right) - \partial_{\bar{z}}\left(A_z - i(\partial_z g)g^{-1}\right)\\ &= \partial_z A_{\bar{z}} - \partial_{\bar{z}}A_z\\ &= F_{z\bar{z}},\end{aligned} \tag{4.30}$$

are invariant under the gauge transformation $g(z,\bar{z}) \in \mathbb{C}^*$. Since $\bar{\psi}$ is a section of the line bundle $\bar{\mathcal{B}}$, we can regard the metric H as a section of the tensor product of the dual bundles, $\bar{\mathcal{B}}^* \otimes \mathcal{B}^*$ [18]. Then the covariant derivatives are given by

$$D_z\psi = \partial_z\psi - iA_z\psi, \tag{4.31}$$

$$D_z\bar{\psi} = \partial_z\bar{\psi} - iA_z\bar{\psi}, \tag{4.32}$$

$$D_z H = \partial_z H + iA_z H - i\bar{A}_{\bar{z}}. \tag{4.33}$$

We are now also able to transform to a holomorphic gauge where $A_{\bar{z}} = 0$ everywhere, and the line bundle $\mathcal{B}$ becomes a holomorphic line bundle [18]. In this holomorphic gauge, the first Bogomolny equation reduces to the Cauchy-Riemann equation $\partial_{\bar{z}}\psi = 0$. Hence, ψ is a holomorphic section of the line bundle $\mathcal{B}$.

In the unitary gauge ($H = 1$) we have that the covariant derivative $D_z H = 0$, so this vanishes in any gauge. Then, in the holomorphic gauge, this is given by

$$D_z H = \partial_z H + iA_z H = 0, \tag{4.34}$$

which has solution

$$A_z = i\partial_z(\log H), \quad A_{\bar{z}} = 0. \tag{4.35}$$

This is known as the Chern connection. The corresponding curvature of the Chern connection is

$$F_{z\bar{z}} = -\partial_{\bar{z}}A_z = -i\partial_z\partial_{\bar{z}}(\log H). \tag{4.36}$$

Hence, the second Bogomolny equation (4.25) simplifies to the Taubes equation, in the form

$$4\partial_z\partial_{\bar{z}}(\log H) + \Omega_M(1 - H|\psi|^2) = 4\pi\sum_{r=1}^{\mathcal{N}}\delta^2(z - Z_r). \tag{4.37}$$

The Gaussian curvature K_M for a general Riemann surface M with metric $ds^2 = \Omega_M(z,\bar{z})dzd\bar{z}$ is given by [21, 32]

$$K_M = -\frac{2}{\Omega_M}\partial_z\partial_{\bar{z}}(\log\Omega_M). \tag{4.38}$$

We are interested in the particular case that the hyperbolic metric $\Omega_M dzd\bar{z}$ has constant Gaussian curvature $K_M = -\frac{1}{2}$, which yields Liouville's equation:

$$4\partial_z\partial_{\bar{z}}(\log\Omega_M) = \Omega_M. \tag{4.39}$$

To construct an $\mathcal{N}$-vortex solution on the Riemann surface M, let N also be a 2-dimensional Riemann surface with local complex coordinate w, metric $\Omega_N(w,\bar{w})$, and curvature $K_N = -\frac{1}{2}$. Then the hyperbolic metric Ω_N satisfies Liouville's equation on N,

$$4\partial_w\partial_{\bar{w}}(\log\Omega_N) = \Omega_N. \tag{4.40}$$

Now, let $f: M \to N$ be a non-constant holomorphic map given locally by the holomorphic function $w = f(z)$, and, working in holomorphic gauge [18], let

$$\psi(z) = \frac{df}{dz}. \tag{4.41}$$

Let the metric H on the holomorphic line bundle $\mathcal{B}$ be the ratio of metrics at z and $f(z)$,

$$H(z, \bar{z}) = \frac{\Omega_N(f(z), \overline{f(z)})}{\Omega_M(z, \bar{z})}. \tag{4.42}$$

Then, away fron any singularities [16] and using Eq. (4.41) and Eq. (4.42), we have that

$$\begin{aligned} 4\partial_z\partial_{\bar{z}}(\log H) &= -4\partial_z\partial_{\bar{z}}(\log \Omega_M) + 4\partial_z\partial_{\bar{z}}(\log \Omega_N) \\ &= -4\partial_z\partial_{\bar{z}}(\log \Omega_M) + 4\partial_w\partial_{\bar{w}}(\log \Omega_N)\frac{df}{dz}\overline{\frac{df}{dz}} \\ &= -\Omega_M + \Omega_N|\psi|^2 \\ &= -\Omega_M\left(1 - H|\psi|^2\right), \end{aligned} \tag{4.43}$$

where we have used the chain rule and the Liouville equations for the metrics, (4.39) and (4.40).

To sumamrize, given the holomorphic map $f: M \to N$, we can construct an $\mathcal{N}$-vortex solution on M by letting our Higgs field be given by

$$|\phi|^2 = \frac{\Omega_N}{\Omega_M}\frac{df}{dz}\overline{\frac{df}{dz}}. \tag{4.44}$$

4.2.1 Rational solutions on $\mathbb{H}^2$

In this section, we work with the Poincaré disk model of $\mathbb{H}^2$, where $\mathbb{H}^2 = \{z \in \mathbb{C} : |z| < 1\}$ and the hyperbolic metric is

$$ds^2 = \Omega_{\mathbb{H}^2}\, dzd\bar{z} = \frac{8}{(1-|z|^2)^2}dzd\bar{z}, \tag{4.45}$$

in which the conformal factor $\Omega_{\mathbb{H}^2}$ satisfies Liouville's equation. We can obtain $\mathcal{N}$-vortex solutions on $\mathbb{H}^2$ by considering the holomorphic mapping $f: \mathbb{H}^2 \to \mathbb{H}^2$, where the target $\mathbb{H}^2$ is also represented by the Poincaré disk model, with complex coordinate w and metric $ds^2 = 8(1-|w|^2)^{-2}dzd\bar{z}$. Then the map f is a holomorphic map from the hyperbolic plane to itself, given by the function $w = f(z)$.

Now, using the results we derived in the previous section, our bundle metric H is thus

$$H(z, \bar{z}) = \frac{(1-|z|^2)^2}{(1-|f(z)|^2)^2} \tag{4.46}$$

and the general solution to the Taubes equation is given by the Higgs field

$$\phi = \frac{1-|z|^2}{1-|f(z)|^2}\frac{df}{dz}, \tag{4.47}$$

with the first Bogomolny equation being satisfied if the Chern connection is

$$A_z = i\partial_{\bar{z}}\log\left(\frac{1-|z|^2}{1-|f(z)|^2}\right). \tag{4.48}$$

On the interior of the disk we require $|f| < 1$ and $|f| = 1$ on the boundary $|z| = 1$, mapping boundary to boundary. To ensure that the boundary condition $|\phi| \to 1$ as $|z| \to 1$ is satisfied, and have finite vortex number $\mathcal{N}$, one needs f to be a Blaschke rational function [18, 24, 32]

$$f(z) = \prod_{m=1}^{\mathcal{N}+1}\frac{z - a_m}{1 - \bar{a}_m z}, \tag{4.49}$$

with the arbitrary complex constants satisfying $|a_m| < 1$. Then the ramification points of the map f are the arbitrary vortex positions in $\mathbb{H}^2$, where the Higgs field ϕ has simple zeros, and the Higgs field takes its vacuum value everywhere on the boundary of the disk. It is worth noting that the topological degree of the mapping is $\deg(f) = \mathcal{N}+1$. However, we can apply a Möbius transformation

$$f \mapsto \frac{f - c}{1 - \bar{c}f}, \tag{4.50}$$

with $|c| < 1$, which leaves our field configuration $\{\phi, A\}$ unchanged up to gauge, and allows us to set $a_{\mathcal{N}+1} = 0$ [24]. Hence, $f(z)$ is now a rational map of the form

$$f(z) = z \prod_{m=1}^{\mathcal{N}} \frac{z - a_m}{1 - \bar{a}_m z}. \tag{4.51}$$

The $\mathcal{N}$-vortex moduli space $\mathcal{M}_{\mathcal{N}}$ has real dimension $2\mathcal{N}$ and is parameterized by the $\mathcal{N}$ complex coefficients a_m, however, these are not natural coordinates on $\mathcal{M}_{\mathcal{N}}$ [24].

The simplest example of an $\mathcal{N}$-vortex solution is given by the map $f(z) = z^{\mathcal{N}+1}$, which is represented by $\mathcal{N}$ vortices coincident at the origin [32]. In this case, the Higgs field is

$$\phi = \frac{1 - |z|^2}{1 - |z|^{2(\mathcal{N}+1)}} (\mathcal{N} + 1) z^{\mathcal{N}} = \frac{(\mathcal{N} + 1) z^{\mathcal{N}}}{1 + |z|^2 + \cdots + |z|^{2\mathcal{N}}}. \tag{4.52}$$

Density plots of this Higgs field $\phi = \phi_1 + i\phi_2$ for various vortex numbers are shown in Table 1. It can be readily seen that the boundary condition $|\phi| \to 1$ is satisfied.

5 Outlook and future work

In a paper of Spruck and Yang [30], they prove the Julia–Zee theorem for a (2 + 1)-dimensional Minkowski spacetime $\mathbb{R} \times \mathbb{R}^2$. It would be interesting to see if there exists a generalization of the Julia–Zee theorem (see Section 2.2) to a spacetime $\mathbb{R} \times M$, where M is a general, compact Riemann surface. We have already shown, in Section 2.2.1, that the temporal gauge field component A_0 satisfies the differential equation (2.76) over M in the static case, thereby recovering the flat case (2.64) by setting $\Omega = 1$. If the Julia–Zee theorem is generalizable to a spacetime $\mathbb{R} \times M$, this would imply that in the static case A_0 is a constant and, in particular, if ϕ is not identically zero then $A_0 = 0$ everywhere on M. The resulting finite-energy static solutions of a (2 + 1)-dimensional spacetime $\mathbb{R} \times M$ would be necessarily electrically neutral.

There are a few variations on the theme that have been considered in the literature, they are briefly detailed below for the reader.

In a recent stimulating paper, Manton [32] deduced a number of vortex equations that are integrable for surfaces of suitable curvature. Consider a Riemann surface M, which may be compact or open with a boundary at spatial infinity, with position-dependent conformal factor Ω_M. Manton considers a two-parameter generalization of the static energy functional in the Abelian Higgs model,

$$E[\phi, A] = \int_M \left\{ \frac{F_{12}^2}{\Omega_M^2} - \frac{2C}{\Omega_M} \left(\overline{D_1\phi} D_1\phi + \overline{D_2\phi} D_2\phi \right) + \left(-C_0 + C|\phi|^2 \right)^2 \right\} \Omega_M \, d^2x, \tag{5.1}$$

where C and C_0 are real constants, which can be rescaled to -1, 0, or 1 [34]. Applying the Bogomolny argument reduces this to

$$E[\phi, A] = -4\pi C_0 \mathcal{N} + \int_M \left\{ \frac{1}{\Omega_M} \left[F_{12} + \Omega_M \left(C_0 - C|\phi|^2 \right) \right]^2 - 2C|D_1\phi + iD_2\phi|^2 \right\} d^2x. \tag{5.2}$$

In this case, the Bogomolny equations are

$$D_1\phi + iD_2\phi = 0 \tag{5.3}$$

$$\frac{1}{\Omega_M} F_{12} = -C_0 + C|\phi|^2. \tag{5.4}$$

Exactly as before, we can use the first Bogomolny equation (5.3) to eliminate the gauge potential. Then by setting $|\phi|^2 = e^{2h}$, away from the zeros of the Higgs field, the Taubes' equation takes the form

$$\nabla^2 h + \Omega_M \left(-C_0 + Ce^{2h} \right) = 0. \tag{5.5}$$

Manton deduces that non-singluar vortex solutions exist for only *five* particular combinations of the constants C_0 and C. They are as follows:

- $C_0 = -1, C = -1$: This yields the Taubes' equation (4.17) (away from the logarithmic singularities of h) that we have been considering throughout this paper;
- $C_0 = 0, C = 1$: This gives the Jackiw–Pi vortex equation [35–37], arising in first order non-relativistic Chern–Simons theory, where $|\phi|^2$ tends to zero at the zeros of the Higgs field ϕ and at spatial infinity (for non-compact surfaces);
- $C_0 = 1, C = 1$: This is the Popov equation for integrable vortices on a 2-sphere of curvature $\frac{1}{2}$ [38, 39], which is a modified version of the Bogomolny equations for $U(1)$ Abelian Higgs vortices [40];
- $C_0 = -1, C = 0$: This corresponds to the Bradlow limit for vortices [26], in which the Bradlow bound (4.13) is saturated exactly and the Higgs field ϕ vanishes everywhere;

- $C_0 = -1, C = 1$: This gives the Ambjørn–Olesen vortex (for $\Omega_M = 1$), in which they studied instabilities in strong electroweak magnetism [41, 42]. The magnetic field strength is enhanced away from the centres of the vortices, illustrating an anti-Meissner effect [32].

In a further paper of Contatto and Dunajski [34], they demonstrate that the five vortex equations detailed above arise as the symmetry reduction of the anti-self-dual Yang–Mills equations in four dimensions.

Maldonaldo and Manton [21] constructed Abelian Higgs vortices on particular compact surfaces of constant negative curvature, whose universal cover has a hyperbolic metric. They demonstrate that these surfaces can be obtained as quotients of the hyperbolic plane.

A more recent paper of Maldonaldo [33] uses $SO(2)$ and $SO(3)$ invariant Yang–Mills instantons to construct Yang–Mills–Higgs monopoles and vortices in hyperbolic space. From this, Maldonaldo is able to determine a direct relation between hyperbolic monopoloes and hyperbolic vortices.

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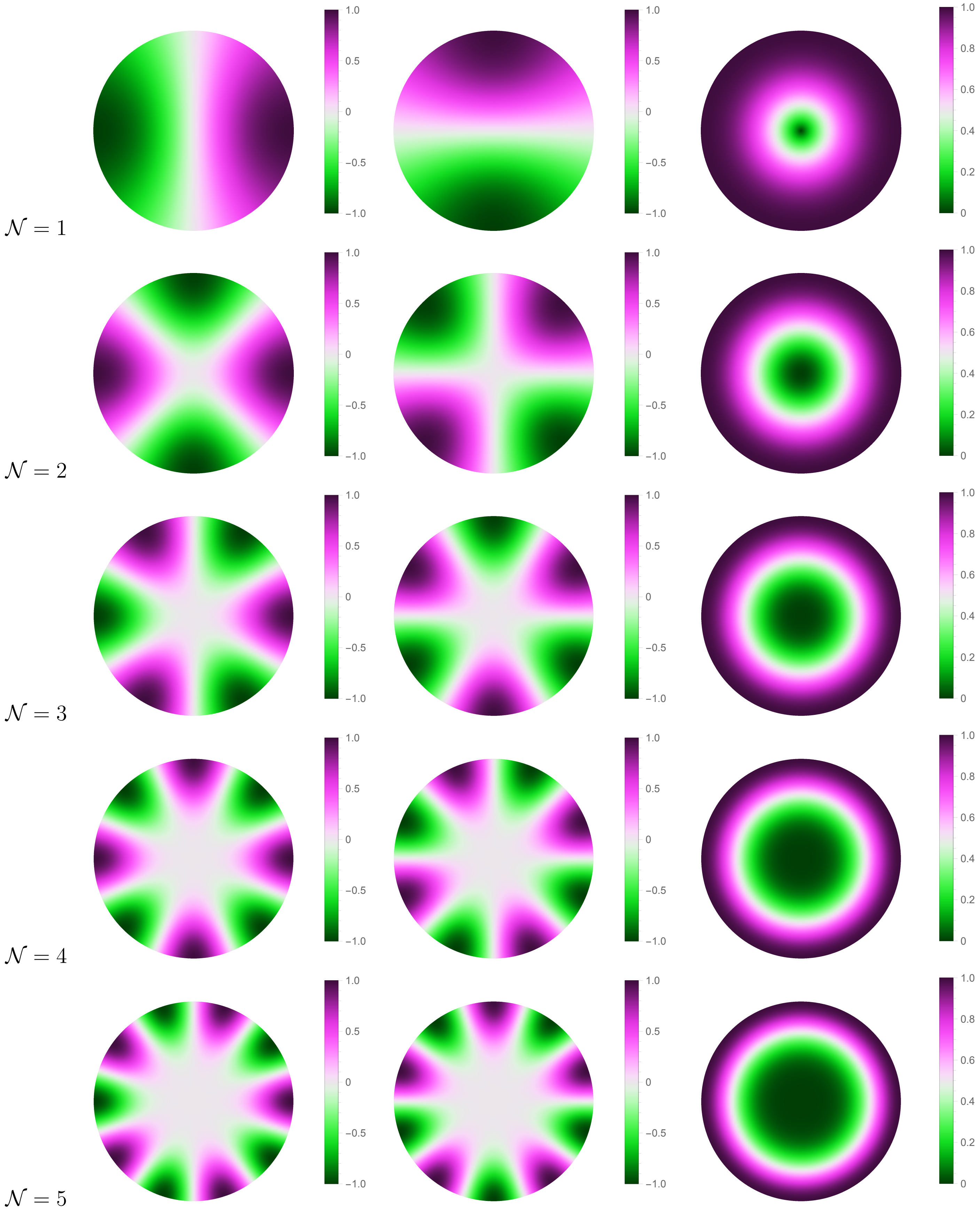

(a) $\phi_1(x^1, x^2)$ (b) $\phi_2(x^1, x^2)$ (c) $|\phi| = \sqrt{|\phi_1|^2 + |\phi_2|^2}$

Table 1: $\mathcal{N}$-vortex solutions corresponding to the holomorphic map $f(z) = z^{\mathcal{N}+1}$. Density plots of (a) $\phi_1(x^1, x^2)$, (b) $\phi_2(x^1, x^2)$, and (c) $|\phi|$, where the Higgs field is $\phi = \phi_1 + i\phi_2$. $\mathcal{N}$-Vortex solutions are shown for $\mathcal{N} \in \{1, 2, 3, 4, 5\}$, where the Higgs field ϕ is given by (4.52).